4/28/09 Math 152 (on 4/30/09 the lecture was all computer examples).

<u>Review:</u> Confidence intervals and Hypothesis testing
(Confidence intervals for the Mann-Whitney test)

<u>e.g.</u> $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$

μ, ~~σ~~ unknown.
σ^2 known

$$H_0: \mu = \mu_0$$

$$H_A: \mu \neq \mu_0.$$

A test at level α rejects for

$$|\bar{X} - \mu_0| > x_0$$

with x_0 chosen so that $P(|\bar{X} - \mu_0| > x_0) = \alpha$.

if H_0 is true.

i.e. $x_0 = \sigma_{\bar{X}} z(\alpha/2)$

where $z(\alpha/2) = \Phi^{-1}(1 - \alpha/2)$

and $\sigma_{\bar{X}} = \sigma/\sqrt{n}$

The test accepts when

$$|\bar{X} - \mu_0| \le \sigma_{\bar{X}} z(\alpha/2)$$

$$\Leftrightarrow \quad -\sigma_{\bar{X}} z(\alpha/2) \le \bar{X} - \mu_0 \le \sigma_{\bar{X}} z(\alpha/2)$$

$$\Leftrightarrow \quad \bar{X} - \sigma_{\bar{X}} z(\alpha/2) \le \mu_0 \le \bar{X} + \sigma_{\bar{X}} z(\alpha/2)$$

and we see that μ_0 lies in the $100(1-\alpha)\%$ confidence interval for μ $\Leftrightarrow$ the hypothesis test accepts.

<u>i.e.</u>

The confidence interval consists exactly of the values of μ_0 for which the Null hypothesis $H_0\ \mu = \mu_0$ is accepted.

More generally

if θ is a parameter of a family of probability distributions and the possible values of θ are Θ

let $\vec{X}$ denote a vector of data from one such distribution.

Suppose that for each $\theta_0 \in \Theta$ there is a test at level α of

$$H_0: \theta = \theta_0.$$

Let $A(\theta_0)$ be the acceptance region of the test (of $H_0: \theta = \theta_0$).

Then

$$C(\vec{X}) = \{\theta : \vec{X} \in A(\theta)\}$$

is $100(1-\alpha)\%$ confidence region for θ.

then

$$A(\theta_0) = \{ \vec{X} \mid \theta_0 \in C(\vec{X}) \}$$

is an acceptance region for a test at level α of

$$H_0 : \theta = \theta_0 .$$

because

$$P(X \in A(\theta_0) \mid \theta = \theta_0) = P(\theta_0 \in C(\vec{X}) \mid \theta = \theta_0)$$
$$= 1 - \alpha .$$

i.e.

The hypothesis that $\theta = \theta_0$ is accepted if θ_0 lies in the confidence region.

In other words

A $100(1-\alpha)\%$ confidence region for θ consists of all values of θ_0 for which the hypothesis $\theta=\theta_0$ is not rejected at level α.

The reason is that

$$P(X \in A(\theta_0) \mid \theta=\theta_0) = 1-\alpha.$$

and

$$P[\theta_0 \in C(X) \mid \theta=\theta_0] = P(X \in A(\theta_0) \mid \theta=\theta_0]$$
$$= 1-\alpha.$$

On the other hand,

if $C(\vec{X})$ is $100(1-\alpha)\%$ confidence region for θ

i.e. for each θ_0.

$$P[\theta_0 \in C(\vec{X}) \mid \theta=\theta_0] = 1-\alpha$$

e.g. Confidence intervals for the Mann Whitney Test

X's $\sim F$

Y's $\sim G$

Suppose $G(x) = F(x - \Delta)$

i.e. the treated group (Y's)

is different from the control group

by an additive effect Δ.

We will find a confidence interval

$(100(1-\alpha)\%)$ for Δ by testing

$H_0: \Delta = \Delta_0$. at level α

and noting which values of Δ_0 allow

H_0 to be accepted.

Recall that

$$U_Y = \#\{(i,j) : (X_i - Y_j) < 0\}$$

was used to test the Null

$$H_0: F = G.$$

Under the Null.

$$H_0: \Delta = \Delta_0 \quad ,$$

$$U_Y(\Delta_0) = \#\{(i,j): \; (X_i - (Y_j - \Delta_0)) < 0\}$$

will have the same distribution as U_Y under $H_0: \Delta = 0$.

$$= \#\{(i,j): \; Y_j - X_i > \Delta_0\}.$$

In particular

$$P\left(U_Y(\Delta_0) = \frac{mn}{2} + k\right) = P\left(U_Y(\Delta_0) = \frac{mn}{2} - k\right)$$

for all integers k.

If $k = k(\alpha)$ is chosen so that

$$P(k \le U_Y(\Delta_0) \le mn - k) = 1 - \alpha$$

the the test at level α accepts $H_0: \Delta = \Delta_0$ for data such that

$$k \le U_Y(\Delta_0) \le mn - k.$$

So a $100(1-\alpha)\%$ confidence interval for Δ is.

$$C = \{\Delta \;;\; k \le U_Y(\Delta) \le mn-k\}.$$

and this is given by:

$$C = [D_{(k)}, D_{(mn-k+1)}]$$

where

the $D_{(1)} \ldots, D_{(m,n)}$

are the ordered differences.

because if $\Delta = D_{(k)}$

$$U_Y(\Delta) = \#(X_i - Y_j + \Delta < 0)$$

$$= \#(Y_j - X_i > \Delta)$$

$$= mn-k.$$

and if $\Delta = D_{(mn-k+1)}$ then

$$U_Y(\Delta) = \#(X_j - X_i > \Delta) = k.$$

Note that a "shift" model for a treatment effect is often overly simplistic!

examples on the computer.