Math 152

3/12/09 and 3/24/09 (Spring Break)

Neyman - Pearson

- H_0, H_1 simple.
- The test which rejects H_0 when likelihood ratio is $< C$ has significance level α.

Then, any other test with sig. level $\leq \alpha$ has power $\leq$ the likelihood ratio test.

Pf: $f(x)$ = density of the observations

$$H_0 : f(x) = f_0(x)$$
$$H_1 \quad f(x) = f_A(x)$$

x takes vector values

A test is a decision function

$d(x)$ s.t $\begin{cases} d(x) = 0 & \text{when } H_0 \text{ accepted} \\ d(x) = 1 & H_0 \text{ rejected} \end{cases}$

$d(X)$ is Bernoulli

$$E(d(X)) = \text{Prob}(d(X) = 1).$$

The significance level of the test is $\alpha = \text{Prob}_{H_0}(d(X)=1) = E_{H_0}(d(X))$.

and the power is $P_A(d(X)=1)$

$$= E_A(d(X)).$$

Suppose the decision function $d(X)$ corresponds to the likelihood ratio test

$$d(x) = 1 \quad \text{if} \quad f_0(x) < c\, f_A(x)$$

and $E_0(d(X)) = \alpha$

Let $d^*(x)$ be the decision function of another test s.t.

$$E_0(d^*(X)) \le E_0(d(X)) = \alpha$$

We claim that

$$E_A(d^*(X)) \le E_A(d(X)).$$

We have

$$d^*(x)\left[c f_A(x) - f_0(x)\right] \leq d(x)\left[c f_A(x) - f_0(x)\right]$$

because if $d(x) = 1$ then $c f_A(x) - f_0(x) > 0$
and if $d(x) = 0$ then $c f_A(x) - f_0(x) \leq 0$.

Integrating or summing this inequality gives

$$c E_A[d^*(X)] - E_0[d^*(X)] \leq c E_A(d(X)) - E_0(d(X))$$

so $0 \leq E_0(d(X)) - E_0[d^*(X)] \leq c\left[E_A(d(X)) - E_A(d^*(X))\right]$

and $\therefore$ $E_A(d^*(X)) \leq E_A(d(X))$.

Uniformly Most Powerful Tests

Notice that in the test of normality

$H_0: (\sigma^2, \mu_0)$ vs. $H_1: (\sigma^2, \mu_1)$

$\mu_1 > \mu_0$.

the most powerful test rejects

for $\bar{X} > x_0$

where x_0 is chosen so that.

$$\frac{x_0 - \mu_0}{\sigma/\sqrt{n}} = \Phi^{-1}(1-\alpha)$$

(Φ is the c.d.f of $N(0,1)$)

so x_0 does not depend on μ_1.

We say that the test is <u>uniformly</u> <u>most</u> powerful against

$H_1: \mu > \mu_0$.

When testing two composite hypotheses such as

$$H_0 : \mu \le \mu_0 \qquad H_1 : \mu > \mu_0$$

We still use a test of the form

$$\bar{X} > X_0 .$$

and have

$$\text{Prob}\{\text{type I error}\} \le \alpha$$

when $\dfrac{X_0 - \mu_0}{\sigma/\sqrt{n}} = \Phi^{-1}(1-\alpha)$.

In such a situation we can still refer to the test as <u>uniformly</u> <u>most powerful</u> against H_1.

But in general, when testing composite hypotheses, there ~~is~~ maybe no uniformly most powerful test.

σ^2 known
$X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$ μ unknown

e.g.

$H_0 : \mu = \mu_0$

$H_A : \mu \neq \mu_0$

Against $H_A : \mu > \mu_0$

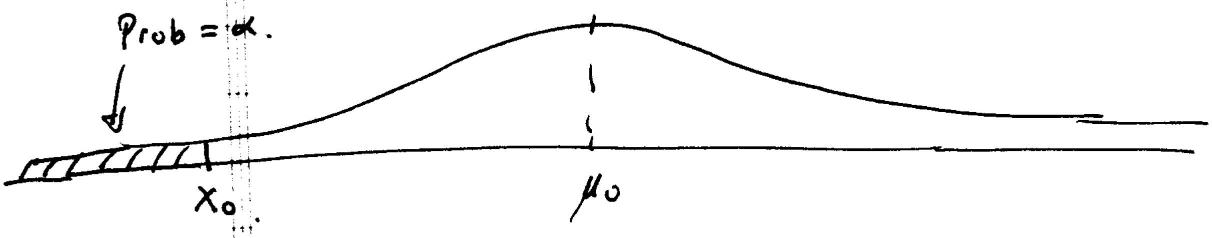
a test which rejects for $\bar{X} > x_0$ is uniformly most powerful.

but against

$H_A : \mu < \mu_0$

the most powerful test is of the form

$\bar{X} < x_0$. (for some x_0.)



So there is no uniformly most powerful test.

We can still construct a rejection region at significance level α for this example.

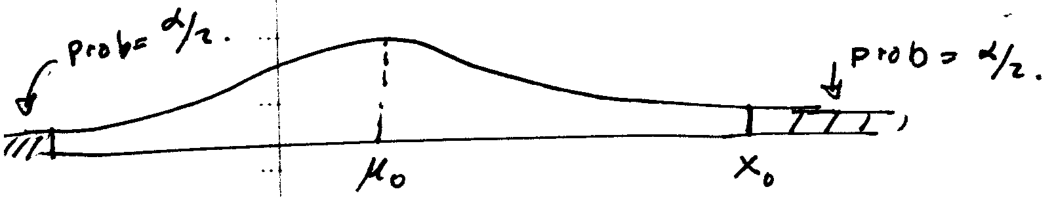
Reject H_0 if $|\bar{X}-\mu_0| > x_0$

where $P(|\bar{X}-\mu_0| > x_0) = \alpha$.

if H_0 holds.

i.e. $x_0 = \sigma_{\bar{X}} \cdot \Phi^{-1}(1-\frac{\alpha}{2})$.

$= \frac{\sigma}{\sqrt{n}} \Phi^{-1}(1-\alpha/2)$.



This test accepts H_0 when

$$|\bar{X}-\mu_0| < \sigma_{\bar{X}} \Phi^{-1}(1-\alpha/2)$$

$$\Leftrightarrow \quad -\sigma_{\bar{X}} \Phi^{-1}(1-\alpha/2) < \bar{X}-\mu_0 < \sigma_{\bar{X}} \Phi^{-1}(1-\tfrac{\alpha}{2})$$

$$\Leftrightarrow \quad \bar{X} - \sigma_{\bar{X}} \Phi^{-1}(1-\tfrac{\alpha}{2}) < \mu_0 < \bar{X} + \sigma_{\bar{X}} \Phi^{-1}(1-\alpha/2)$$

and $\left(\bar{X} - \sigma_{\bar{X}} \Phi^{-1}(1-\frac{\alpha}{2}),\ \bar{X} + \sigma_{\bar{X}} \Phi^{-1}(1-\alpha/2)\right)$

is a $(1-\alpha)$ confidence interval

We see that the confidence interval consists of all values of μ_0 for which the Hypothesis $H_0: \mu = \mu_0$ would be accepted at level α.

More generally,

if θ is a parameter for a family of of distributions

Let Θ be the set of possible θ

X be the data vector

Thm: Suppose that for every $\theta_0 \in \Theta$

there is a test of $H_0: \theta = \theta_0$

with sig. level α.

which accepts H_0 for all $X \in A(\theta_0)$

Then $C(\vec{X}) = \{\theta : X \in A(\theta)\}$

is $(1-\alpha)$ confidence region for θ.

Pf: $P\left(X \in A(\theta_0) \mid \theta=\theta_0\right) = 1-\alpha$

$P(\theta_0 \in C(X) \mid \theta = \theta_0)$

$= P(X \in A(\theta_0) \mid \theta=\theta_0) = 1-\alpha.$

Thm: Values of θ for which the hypothesis $\theta=\theta_0$ will not be rejected at level α (with the given data) form a $1-\alpha$ confidence region for θ.

<u>ThmB</u>. If $C(\vec{X})$ is $1-\alpha$ confidence region for θ

i.e. $P(\theta_0 \in C(X) \mid \theta=\theta_0) = 1-\alpha.$

then

$A(\theta_0) = \{\vec{X} : \theta_0 \in C(\vec{X})\}$ is an acceptance region for $H_0: \theta=\theta_0$ at level α.

be cause

$$P(X \in A(\theta_0) \mid \theta = \theta_0)$$

$$= \text{Prob}(\theta_0 \in C(X) \mid \theta = \theta_0) = 1 - \alpha.$$

$\boxtimes$

#1 ten tosses of a coin

$H_0: p = 1/2$

$H_1: p \neq 1/2.$

reject H_0 if 0 or 10 heads observed.

a) significance level?.

= Prob(reject H_0 when it is true).

= Prob(0 heads or 10 heads with $p = 1/2$).

$= 1/512.$

b) if (in fact) $p = .1$, what is the power?

Prob(reject H_0 when false (and $p = .1$))

$$= (.9)^{10} + (.1)^{10} \approx .349$$

#3. $X \sim bin(100, p)$

$H_0: p = .5$

$H_A: p \neq .5$

test rejects H_0 if $|X-50| > 10$.

a) What is α?.

$$\text{Prob}(|X-50| > 10 \mid p = .5)$$

$$= \text{Prob}\left(\left|\frac{X-50}{\sqrt{100 \cdot \frac{1}{2}(1-\frac{1}{2})}}\right| > \frac{10}{5}\right)$$

$$\approx 1 - \text{Prob}(-2 < Y < 2) \quad \text{where } Y \sim N(0,1).$$

$$\approx ~~\text{with}~~\ 2(1 - .9772) = 2(.0228)$$

$$= .0456$$

b) graph power as a function of p.

If the "true" parameter is p.

$$\cancel{\text{Prob}\left(\left|\frac{X - 100p}{\sqrt{100p(1-p)}}\right|\right)}$$

$$\text{Prob}(-10 < X - 50 < 10) = \text{Prob}(40 < X < 6$$

$$= \text{Prob}\left(\frac{40-100p}{\sqrt{100p(1-p)}} < \frac{X-100p}{\sqrt{100p(1-p)}} < \frac{60-100p}{\sqrt{100p(1-p)}} \right)$$

$$\approx \frac{1}{\sqrt{2\pi}} \int_{\frac{40-100p}{\sqrt{100p(1-p)}}}^{\frac{60-100p}{\sqrt{100p(1-p)}}} e^{-x^2/2}\, dx.$$

$$\text{power}(p) = 1 - \frac{1}{\sqrt{2\pi}} \int_{\frac{40-100p}{\sqrt{100p(1-p)}}}^{\frac{60-100p}{\sqrt{100p(1-p)}}} e^{-x^2/2}\, dx.$$

$$= 1 - \Phi\left(\frac{60-100p}{\sqrt{100p(1-p)}}\right) + \Phi\left(\frac{40-100p}{\sqrt{100p(1-p)}}\right)$$

#5 a, b, c, d, e. true-false

~~#7 $X_1, \ldots, X_n$ sampled from Poisson~~

#7 $X_1, \ldots, X_n$ sample from Poisson

$H_0: \lambda = \lambda_0 \qquad H_1: \lambda = \lambda_1 \qquad \lambda_1 > \lambda_0.$

$$\frac{\prod_{i=1}^{n} \left(\frac{\lambda_0^{X_i}}{X_i!} e^{-\lambda_0} \right)}{\prod \frac{\lambda_1^{X_i}}{X_i!} e^{-\lambda_1}} = \left(\frac{\lambda_0}{\lambda_1}\right)^{\sum X_i} e^{n(\lambda_1 - \lambda_0)}$$

#7 cont. Rejection (of H_0) Region

has form

$$\left(\frac{\lambda_0}{\lambda_1}\right)^{\Sigma X_i} e^{n(\lambda_1-\lambda_0)} < X_0 .$$

Since $\left(\frac{\lambda_0}{\lambda_1}\right) < 1$ this corresponds

to a large value of $\sum_{i=1}^{n} X_i$

So we reject when

$$\sum_{i=1}^{n} X_i > N(\alpha) .$$

For significance level α we need $N(\alpha)$ s.t.

$$P\left(\sum_{i=1}^{n} X_i > N(\alpha) \,\Big|\, H_0 \right) \le \alpha .$$

Since $\sum_{i=1}^{n} X_i$ is Poisson $n\lambda_0$ if H_0 holds

we choose $N(\alpha)$ s.t.

$$\sum_{k=N(\alpha)}^{\infty} \frac{(n\lambda_0)^k}{k!} e^{-n\lambda_0} \le \alpha$$

or $$\sum_{n=0}^{N(\alpha)-1} \frac{(n\lambda_0)^k}{k!} e^{-n\lambda_0} > \alpha .$$

#9 $X_1, \dots X_{25}$ $N(?, 100)$

find rejection region: $\alpha = .10$

$H_0 : \mu = 0$

$H_A : \mu = 1.5$

What is the power?

repeat for $\alpha = .01$.

Simple Hypotheses

H_0 is rejected (for small values of.) for

$$\frac{\cancel{\left(\frac{1}{\sqrt{2\pi}\,10}\right)^n} e^{-\frac{1}{2}\sum_{i=1}^{25}\frac{X_i^2}{100}}}{\cancel{\left(\frac{1}{\sqrt{2\pi}\cdot 10}\right)^n} e^{-\frac{1}{2}\sum_{i=1}^{25}\frac{(X_i-1.5)^2}{100}}} < X_0.$$

i.e. for large values of.

$$\sum_{i=1}^{25}\left[\frac{X_i^2}{100} - \frac{(X_i-1.5)^2}{100}\right].$$

$$= \frac{1}{100}\sum_{i=1}^{25}(X_i-(X_i-1.5))(X_i+(X_i-1.5)).$$

$$= \frac{1.5}{100}\sum_{i=1}^{25}(2X_i-1.5)$$

i.e. for large values of $\frac{1}{25}\sum_{i=1}^{25} X_i$.

#9 cont.

If H_0 holds.

$$\text{Prob}\left(\frac{\bar{X}}{10/5} > \frac{x_0}{10/5}\right) = 1 - \Phi\left(\frac{x_0}{2}\right).$$

$$1 - \Phi\left(\frac{x_0}{2}\right) = .10.$$

$$\Rightarrow \Phi\left(\frac{x_0}{2}\right) = .9.$$

$$\Rightarrow \frac{x_0}{2} \approx 1.28$$

$$\Rightarrow x_0 \approx 2.56.$$

reject if $\bar{X} > 2.56$.

If H_1 holds then

$$P(\bar{X} > 2.56.)$$

$$= P\left(\frac{\bar{X} - 1.5}{2} > \frac{2.56 - 1.5}{2}\right)$$

$$= P\left(\frac{\bar{X} - 1.5}{2} > .53\right).$$

$$= 1 - \Phi(.53) \approx 1 - .7019$$

$$\cong .2981$$

9 cont.

for $\alpha = .01$.

$$1 - \Phi\left(\frac{x_0}{2}\right) = .01$$

$$\Phi\left(\frac{x_0}{2}\right) = .99$$

$$\frac{x_0}{2} \simeq 2.33$$

$$x_0 \simeq 4.66.$$

reject if $\bar{X} > 4.66$.

power:

$$P(\bar{X} > 4.66)$$

$$= P\left(\frac{\bar{X} - 1.5}{2} > \frac{4.66 - 1.5}{2}\right)$$

$$= P\left(\frac{\bar{X} - 1.5}{2} > 1.58\right)$$

$$= 1 - \Phi(1.58) \simeq 1 - .9429$$

$$\simeq .0571$$

#12 $X_1, \dots, X_n$ random sample exponential.

$$f(x/\theta) = \theta e^{-\theta x}$$

$H_0 : \theta = \theta_0$.

$H_A \; \theta \neq \theta_0$.

$$\Lambda = \frac{\theta_0^n e^{-\theta_0(\Sigma X_i)}}{(\text{max likelihood})}.$$

$$l(\theta) = n \log \theta - \theta \Sigma X_i.$$

$$l'(\theta) = \frac{n}{\theta} - \Sigma X_i = 0.$$

$$\hat{\theta} = \frac{1}{\overline{X}}$$

$$\Lambda = \frac{\theta_0^n e^{-\theta_0 n \overline{X}}}{(\frac{1}{\overline{X}})^n e^{-\frac{1}{\overline{X}} n \overline{X}}} = \theta_0^n (\overline{X})^n e^{-\theta_0 n \overline{X}} e^n$$

$$= \theta_0^n (\overline{x} e^{-\theta_0 \overline{X}})^n e^n$$

The likelihood ratio test rejects for

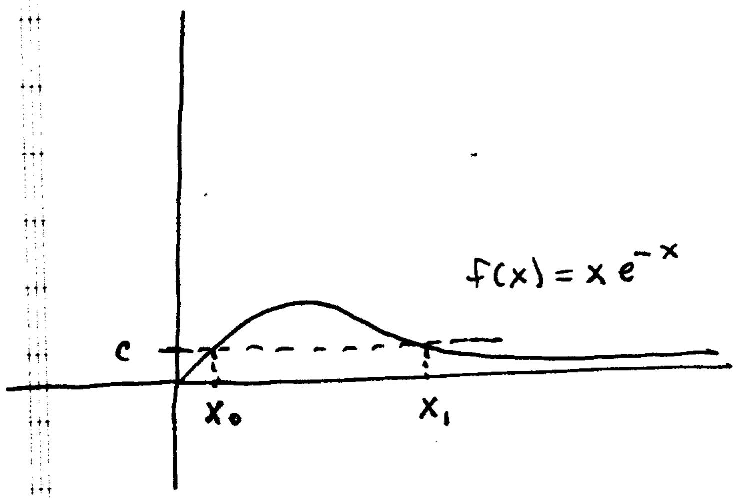
$$\{ \overline{X} e^{-\theta_0 \overline{X}} < c \}.$$

#13. Same set up as 12.

$\theta_0 = 1$, $n = 10$, $\alpha = .05$.

a) Rejection region is

$$\{\bar{X} e^{-\bar{X}} \le c\}.$$



which corresponds to

$$\{\bar{X} \le x_0\} \cup \{\bar{X} \ge x_1\}$$

where x_0, x_1 are solutions of

$$x e^{-x} = c.$$

b) For significance level $\alpha = .05$

Choose c s.t

$$\text{Prob}\{\bar{X} e^{-\bar{X}} \le c\} = .05.$$

c) $$E\left(e^{t \sum_{i=1}^{10} X_i}\right) = E\left(e^{t X_1}\right)^{10} = \left(\frac{1}{1-t}\right)^{10}$$

$$\int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{(t-1)x} dx = \frac{1}{1-t}, \quad t < 1.$$

which is the m.g.f of.

$$\Gamma(\lambda = 1, \alpha = 10.)$$

So $f_{(\Sigma X_i)}(x) = \frac{1}{\Gamma(10)} x^9 e^{-x}$

Generalized Likelihood Ratio Tests

$\vec{X} = (X_1, \ldots, X_n)$

have joint dist

$f(\vec{x} \mid \theta)$

$H_0: \quad \theta \in \omega_0.$

$\omega_0 \cap \omega_1 = \emptyset.$

$H: \quad \theta \in \omega_1$

let $\Omega = \omega_0 \cup \omega_1$

Let

$$\Lambda^* \equiv \frac{\max_{\theta \in \omega_0} [\operatorname{lik}(\theta)]}{\max_{\theta \in \omega_1} [\operatorname{lik}(\theta)]}$$

then small values of Λ^*

discredit H_0.

a more convenient statistic is

$$\Lambda \equiv \frac{\max_{\theta \in \omega_0} [\operatorname{lik}(\theta)]}{\max_{\theta \in \Omega} [\operatorname{lik}(\theta)]}.$$

Note that $\Lambda = 1$ if $\Lambda^* \geq 1$

and $\Lambda = \Lambda^*$ if $\Lambda^* \leq 1$.

A likelihood ratio test will have a rejection region of the form

$$\Lambda \leq \lambda_0.$$

where $P(\Lambda \leq \lambda_0 \mid H_0) = \alpha$.

is the significance level.