

Midterm II Solutions

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1. Suppose that in a city, the number of suicides can be approximated by a Poisson process with $\lambda = .33$ per month.

- (a) Find the probability of k suicides in a year for $k = 1, 2, \dots$
 What is the most probable number of suicides?
 (b) What is the probability of two suicides in one week?

a) $12 (.33) = 3.96$

$$p_k = \frac{(3.96)^k}{k!} e^{-3.96}$$

$$\frac{p_{k+1}}{p_k} = \frac{3.96}{k+1} \quad p_4 < p_3 \\ p_0 < p_1 < p_2 < p_3$$

p_3 is largest, $k=3$ is the most probable number

b) $\text{Prob}(2 \text{ in one week}) \approx \left(\frac{.33}{4}\right)^2 e^{-.33/4}$
 (≈ 4 weeks in a month).

2. Find the density function of $Y = e^Z$, where $Z \sim N(\mu, \sigma^2)$.

$$\begin{aligned} \text{Prob} \{ e^z \leq y \} &= 0 \quad y \leq 0 \\ &= \text{Prob} \{ z \leq \log y \} \quad y > 0. \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\log y} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}} \cdot \frac{1}{y}.$$

3. If X and Y are independent random variables, find $\text{Var}(XY)$ in terms of the means and variances of X and Y .

$$\begin{aligned}
 \text{Var}(XY) &= E((XY)^2) - E(XY)^2 \\
 &= E(X^2)E(Y^2) - E(X)^2E(Y)^2 \\
 &\quad (\text{by independence}) \\
 &= (\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2) - \mu_x^2\mu_y^2 \\
 &= \sigma_x^2\sigma_y^2 + \mu_x^2\sigma_y^2 + \sigma_x^2\mu_y^2
 \end{aligned}$$

4. A random square has a side length which is uniform on $[0, 1]$.
Find the expected area of the square.

$$\int_0^1 u^2 du = \frac{1}{3} u^3 \Big|_0^1 = \underline{\underline{\frac{1}{3}}}$$

5. Let X be a Poisson random variable with parameter $\lambda > 0$. Find $E(e^{tX})$ as a function of the real variable t . Show how to use this function to compute the variance of the Poisson random variable.

$$\begin{aligned} E(e^{tX}) &= \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{t+\lambda})^k}{k!} = e^{-\lambda} e^{t+\lambda} \\ &= e^{\lambda(e^{t+\lambda}-1)} \end{aligned}$$

$$\left. \frac{d}{dt} (e^{\lambda(e^{t+\lambda}-1)}) \right|_{t=0} = \left. \lambda e^t e^{\lambda(e^{t+\lambda}-1)} \right|_{t=0} = \lambda.$$

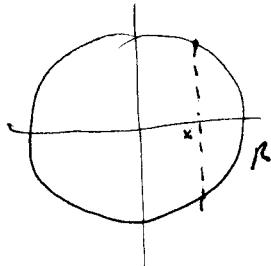
$$\begin{aligned} \left. \frac{d^2}{dt^2} (\lambda e^t e^{\lambda(e^{t+\lambda}-1)}) \right|_{t=0} &= \left. \left(\lambda e^t e^{\lambda(e^{t+\lambda}-1)} + \lambda^2 e^{2t} e^{\lambda(e^{t+\lambda}-1)} \right) \right|_{t=0} \\ &= \lambda + \lambda^2. \end{aligned}$$

$$\text{Var}(X) = \lambda + \lambda^2 - (\lambda)^2 = \lambda.$$

6. Let X and Y be the x and y coordinates respectively of a point chosen uniformly at random in a circle of radius R centered at the origin.

- (a) Find the joint density of X and Y .
- (b) Find the marginal densities of X and Y .
- (c) Find the conditional density of X given Y .
- (d) Find the probability that the distance to the origin of the point selected is not greater than a .

$$a) \quad f_{XY}(x, y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\ 0 & \text{else} \end{cases}$$



$$b) \quad f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2} \quad \text{by symmetry.}$$

$$c) \quad f_{X|Y}(x|y) = \frac{\frac{1}{\pi R^2}}{\frac{2\sqrt{R^2-y^2}}{\pi R^2}} = \frac{1}{2\sqrt{R^2-y^2}}, \quad -\sqrt{R^2-y^2} \leq x \leq \sqrt{R^2-y^2}$$

$$= 0 \quad \text{else.}$$

$$d) \quad \frac{\pi a^2}{\pi R^2} = \left(\frac{a}{R}\right)^2$$

7. Find the PDF of $Z = X + Y$, when X and Y are independent exponential random variables with common parameter λ .

$$\begin{aligned}
 f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\
 &= \int_0^{\infty} \lambda e^{-\lambda x} f_Y(z-x) dx \\
 &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\
 &= \lambda^2 e^{-\lambda z} \cdot z, \quad z \geq 0.
 \end{aligned}$$

8. Let X and Y have the joint density

$$f(x, y) = e^{-y}, \quad 0 \leq x \leq y$$

(a) Find $E(X|Y = y)$

(b) Find the density function of the random variable $E(X|Y)$.

$$f_{X|Y}(x|y) = \frac{f_{xy}(x, y)}{f_y(y)}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^y e^{-y} dx = y e^{-y} \quad y \geq 0.$$

$$= 0 \quad \text{else.}$$

$$f_{X|Y}(x|y) = \frac{e^{-y}}{y e^{-y}} = \frac{1}{y} \quad 0 \leq x \leq y.$$

$$= 0 \quad \text{else.}$$

$$a) \quad E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx = \int_0^y x \cdot \frac{1}{y} dx = \frac{1}{y} \frac{y^2}{2} = \frac{y}{2}$$

$$E(X|Y) = \frac{Y}{2}.$$

$$\text{Prob } \left\{ \frac{Y}{2} \leq u \right\} = F_Y(u)$$

$$b). \quad f_{\left(\frac{Y}{2}\right)}(u) = 2 F_Y'(u) = 2 f_Y(u) = 4u e^{-2u} \quad u \geq 0$$

$$= 0 \quad \text{else.}$$