Math 151—Probability.

4:15 PM SECTION

TUESDAY, DEC. 16, 2008

Name: Solutions

Directions: No books or notes are allowed. You may use a calculator but can get full credit without one. A table of values of the CDF of a standard normal distribution is included.

Question	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

1. We draw the top 7 cards form a well-shuffled standard 52-card deck. Find the probability that:
   (a) The 7 cards include exactly 3 aces.
   (b) The 7 cards include exactly 2 kings.
   (c) The probability that the 7 cards include exactly 3 aces, or exactly 2 kings, or both.

a) $$\frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}$$

b) $$\frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}$$

c) $$\frac{\binom{4}{3}\binom{48}{4} + \binom{4}{2}\binom{48}{5} - \binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}$$

2. An internet service provider uses 50 modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independent of the other customers.
   (a) What is the PMF of the number of modems in use at the given time?
   (b) Repeat part 9a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.
   (c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact, as well as an approximate formula based on the Poisson approximation of part (b).

a) X = # of modems in use.

$$\text{Prob}\{X=k\} = \binom{1000}{k}(.01)^k(.99)^{1000-k} \qquad 0 \le k \le 49.$$

$$\text{Prob}\{X=50\} = \sum_{k=50}^{1000}\binom{1000}{k}(.01)^k(.99)^{1000-k}$$

$$\text{Prob}\{X>50\} = 0. = P\{X<0\}.$$

b). Y = # of customers needing a connection is binom$(1000, .01)$.

$$E(Y) = (.01)(1000) = 10.$$

so Y is approximately Poisson w/ parameter $\lambda = 10$.

$$\text{Prob}\{X=k\} = \text{Prob}\{Y=k\} \approx \frac{10^k}{k!}e^{-10} \qquad 0 \le k \le 49.$$

$$\text{Prob}\{X=50\} \approx 1 - \sum_{k=0}^{49}\frac{10^k}{k!}e^{-10}$$

c) $$\text{Prob}\{Y>50\} = \sum_{k=51}^{1000}\binom{1000}{k}(.01)^k(.99)^{1000-k}$$

$$\approx 1 - \sum_{k=0}^{50}\frac{10^k}{k!}e^{-10}$$

3. Let X and Y be normal random variables with means 0 and 1, respectively, and variances 1 and 4, respectively.
   (a) Find $P(X \le 1.5)$ and $P(X \le -1)$.
   (b) Find the PDF of $(Y-1)/2$.
   (c) Find $P(-1 \le Y \le 1)$

a) $X \sim N(0,1)$ $\quad Y \sim N(1,4)$

$P(X \le 1.5) \simeq .9332$ from table.

$P(X \le -1) = 1 - P(X \le 1) \cong 1 - .8413$

$= .1587$

b) $Y^* = \dfrac{Y-1}{2} \sim N(0,1)$

so the p.d.f. is $f_{Y^*}(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

c) $P(-1 \le Y \le 1)$

$= P\left(\frac{-1-1}{2} \le \frac{Y-1}{2} \le \frac{1-1}{2}\right)$

$= P\left(-1 \le \frac{Y-1}{2} \le 0\right)$

$= P(X \le 1) - 1/2$

$\cong .8413 - .5 = .3413$

4. Let X and Y have the joint density function

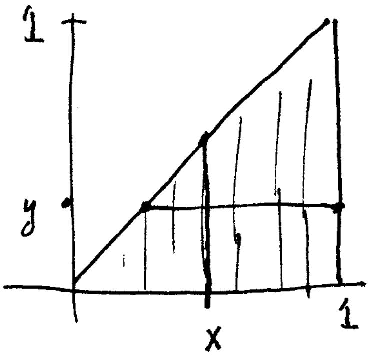
$$f(x,y) = k(x-y), \qquad 0 \le y \le x \le 1$$

and 0 elsewhere.

(a) Find k.

(b) Find the marginal densities of X and Y.

(c) Find the conditional densities of Y given X and X given Y.



$$k \int_0^1 \int_y^1 (x-y)\,dx\,dy = 1.$$

$$= k \int_0^1 \left(\frac{x^2}{2} - yx\right)\Big|_y^1 \, dy$$

$$= k \int_0^1 \left(\frac{1}{2} - y\right) - \left(\frac{y^2}{2} - y^2\right) \, dy$$

$$= k \int_0^1 \left(\frac{1}{2} - y + \frac{y^2}{2}\right) dy$$

$$= k \cdot \left(\frac{1}{2} - \frac{1}{2} + 1/6\right) = 1$$

So $\boxed{k = 6.}$

b)

$$f_X(x) = 6\int_0^x (x-y)\,dy = 6\left(x^2 - \frac{x^2}{2}\right) = 3x^2 \quad 0 \le x \le 1$$

$$f_Y(y) = 6\int_y^1 (x-y)\,dx = 6\left(\frac{x^2}{2} - yx\Big|_y^1\right) = 6\left(\left(\frac{1}{2} - y\right) - \left(\frac{y^2}{2} - y^2\right)\right)$$

$$= 6\left(\frac{1}{2} - y + y^2/2\right)$$

$$= 3y^2 - 6y + 3$$

c)

$$f_{Y|X}(y|x) = \frac{6(x-y)}{6(1/2 - y + y^2/2)} = \frac{x-y}{\frac{1}{2} - y + y^2/2} \quad 0 \le y \le x \le 1.$$

$$f_{X|Y}(x|y) = \frac{6(x-y)}{3x^2} = \frac{2(x-y)}{x^2} \quad 0 \le y \le x \le 1.$$

5. (a) Show that the moment generating function of a geometric random variable with parameter p is

$$\frac{pe^t}{1-(1-p)e^t}$$

You will need to sum an appropriate geometric series.

(b) Using the result of part (a), and moment generating functions, show that if N is geometric with parameter p and S is a sum of N independent exponential random variables with parameter λ, then S exponential with parameter $p\lambda$.

a) X is geometric, $\text{Prob}\{X=k\} = (1-p)^{k-1} \cdot p$

$$E(e^{tX}) = \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1} \cdot p = \frac{p}{1-p} \sum_{k=1}^{\infty} (e^t(1-p))^k$$

$$= \frac{e^t(1-p) \cdot p}{1-p} \sum_{k=0}^{\infty} (e^t(1-p))^k$$

$$= \frac{e^t \cdot p}{1-e^t(1-p)}$$

b) $X_i \sim \lambda e^{-\lambda x}$ independent.

$S = \sum_{i=1}^{N} X_i$ $\qquad E(e^{tS}) = E(E(e^{tS} | N))$

$$E(e^{tS} | N=n) = \left(\frac{\lambda}{\lambda-t}\right)^n$$

$$E(e^{tS} | N) = \left(\frac{\lambda}{\lambda-t}\right)^N$$

$$E\left(\left(\frac{\lambda}{\lambda-t}\right)^N\right) = E\left(e^{N\log\left(\frac{\lambda}{\lambda-t}\right)}\right) = \frac{p\, e^{\log\left(\frac{\lambda}{\lambda-t}\right)}}{1-(1-p)e^{\log\left(\frac{\lambda}{\lambda-t}\right)}} = \frac{\frac{p\lambda}{\lambda-t}}{1-(1-p)\frac{\lambda}{\lambda-t}}$$

$$= \frac{p\lambda}{\lambda-t-(1-p)\lambda} = \frac{p\lambda}{p\lambda-t}$$

6. Suppose that $X_1, X_2, \ldots, X_{20}$ are independent random variables with density functions

$$f(x) = 2x, \qquad 0 \le x \le 1$$

Let $S = X_1 + \cdots + X_{20}$. Use the central limit theorem to approximate $P(S \le 10)$.

$X_i \sim 2x \qquad 0 \le x \le 1.$

$E(X_i) = \int_0^1 2x^2\,dx = \frac{2}{3}$

$E(X_i^2) = \int_0^1 2x^3\,dx = 1/2. \Rightarrow Var(X_i) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

C.L.T $\Rightarrow \quad \dfrac{S - 20\left(\frac{2}{3}\right)}{\sqrt{20/18}} \sim N(0,1).$

$P(S \le 10) = P\left(\dfrac{S - 40/3}{\sqrt{10}/3} \le \dfrac{10 - \frac{40}{3}}{\sqrt{10}/3}\right)$

$= P\left(N(0,1) \le \dfrac{-10/3}{\sqrt{10}/3}\right)$

$= P\left(N(0,1) \le -\sqrt{10}\right)$

$\simeq P\left(N(0,1) \le -3.16\right)$

$\simeq 1 - .9992 = .0008$

7. A particle of mass m has a random velocity, V, which is normally distributed with parameters $\mu = 0$, and σ. Find the density function of the kinetic energy, $E = \frac{1}{2}mV^2$.

$$F_E(x) = \text{Prob}\left\{ \tfrac{1}{2} m V^2 \le x \right\}$$

$$= \text{Prob}\left\{ V^2 \le \frac{2x}{m} \right\} = \text{Prob}\left\{ -\sqrt{\frac{2x}{m}} \le V \le \sqrt{\frac{2x}{m}} \right\}$$

$$= 2\,\text{Prob}\left\{ 0 \le V \le \sqrt{\frac{2x}{m}} \right\}$$

$$= 2\left(F_V\left(\sqrt{\frac{2x}{m}}\right) - \frac{1}{2} \right).$$

$$f_E(x) = 2\, F_V'\left(\sqrt{\frac{2x}{m}}\right) \cdot \left(\sqrt{\frac{2}{m}}\right) \frac{1}{2} \frac{1}{\sqrt{x}} = \sqrt{\frac{2}{m}} \frac{1}{\sqrt{x}} f_V\left(\sqrt{\frac{2x}{m}}\right)$$

$$= \sqrt{\frac{2}{m}} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\left(\sqrt{\frac{2x}{m}}\right)^2}{2\sigma^2}}$$

$$= \sqrt{\frac{2}{m}} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\cancel{2}x}{m\sigma^2}}, \quad x \ge 0$$

$$= \frac{1}{\sqrt{m\pi x}} \frac{1}{\sigma} e^{-\frac{x}{m\sigma^2}}$$

$$x \ge 0$$

8. A child types the letters Q,W,E,R,T,Y randomly producing 1000 letters in all. What is the expected number of times that the sequence "TRY" appears? Should we be surprised if it occurred 100 times? (hint: recall that Markov's inequality for non-negative random variables X says that $P(X > t) \leq E(X)/t$.)

For ten points extra credit, here is a problem of the same variety: Suppose that n enemy aircraft are shot at simultaneously by m gunners, that each gunner selects an aircraft at random to shoot at independently of the other gunners, and that each gunner hits the selected aircraft with probability p. Find the expected number of aircraft hit by the gunners. → done in class.

Let $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ letter is T, } (i+1)^{st} \text{ is R and } (i+2)^{nd} \text{ is Y} \\ 0 & \text{else} \end{cases}$ for $i = 1, \ldots 998$

$S = \sum_{i=1}^{998} X_i$ is the # of occurrences of TRY.

$E(S) = 998\, E(X_i)$

since all positions are equally likely to have "TRY".

$E(X_i) = \frac{1}{6^3} \cdot 1 + (1 - \frac{1}{6^3}) \cdot 0 = \frac{1}{6^3}$

so $E(S) = \frac{998}{6^3}$

$P(\cancel{X} > 99) \leq \frac{998/6^3}{99} \approx \frac{10}{6^3} = \frac{5}{3 \cdot 6 \cdot 6} = \frac{5}{108} \approx \frac{1}{22}$

"surprising".

9. Let T be an exponential random variable, and conditional on T, let U be uniform on $[0, T]$. Find the unconditional mean and variance of U. Recall that

$$\mathrm{Var}(U) = E(\mathrm{Var}(U|T)) + \mathrm{Var}(E(U|T))$$

and that

$$\mathrm{Var}(U|T=t) = E(U^2|T=t) - E(U|T=t)^2.$$

so $E(T^2) = 2/\lambda^2$.

$\mathrm{Var}(T) = \frac{1}{\lambda^2} = E(T^2) - E(T)^2$

$E(T) = \frac{1}{\lambda}$

$T \sim \lambda e^{-\lambda t}, \quad t \geq 0$

$$E(U) = E(E(U|T))$$

$$E(U|T=t) = t/2 \implies E(U|T) = T/2$$

$$E(U) = E(T/2) = \frac{1}{2}E(T) = \frac{1}{2\lambda}$$

$$\mathrm{Var}(U|T=t) = E(U^2|T=t) - E(U|T=t)^2$$

$$= \frac{1}{t}\int_0^t u^2\,du - \left(\frac{t}{2}\right)^2$$

$$= \frac{1}{3}t^2 - \frac{1}{4}t^2 = \frac{1}{12}t^2$$

so

$$\mathrm{Var}(U|T) = \frac{1}{12}T^2; \quad E(\mathrm{Var}(U|T)) = \frac{1}{12}E(T^2) = \frac{1}{12}\left(\frac{2}{\lambda^2}\right)$$

and $\mathrm{Var}(E(U|T)) = \mathrm{Var}(T/2) = \frac{1}{4}\cdot\frac{1}{\lambda^2}$

$$\therefore \quad \mathrm{Var}(U) = \frac{1}{6\lambda^2} + \frac{1}{4\lambda^2} = \frac{5}{12\lambda^2}$$

10. Let X have variance σ^2 and write $m_k = E(X^k)$. Define the *skewness* of X by

$$\text{skw}(X) = \frac{E[(X - m_1)^3]}{\sigma^3}.$$

Show that when X is Poisson with parameter λ,

$$\text{skw}(X) = 1/\sqrt{\lambda}.$$

$E(X) = \lambda \qquad Var(X) = \lambda$

let $X^* = \dfrac{X - \lambda}{\sqrt{\lambda}} = \dfrac{1}{\sqrt{\lambda}} X - \sqrt{\lambda}$

we want $E((X^*)^3)$

$$M_{X^*}(t) = e^{-\sqrt{\lambda} t} M_X\left(\frac{t}{\sqrt{\lambda}}\right)$$

$$= e^{-\sqrt{\lambda} t} e^{\lambda(e^{t/\sqrt{\lambda}} - 1)}$$

$$= e^{\lambda(e^{t/\sqrt{\lambda}} - t/\sqrt{\lambda} - 1)}$$

$$= e^{\lambda\left((t/\sqrt{\lambda})^2/2 + \frac{(t/\sqrt{\lambda})^3}{3!} + \dots\right)}$$

$$= e^{\left(t^2/2 + \frac{t^3}{6\sqrt{\lambda}} + \dots\right)}$$

$$= 1 + \left(\frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \dots\right) + O(t^4)$$

So $M_{X^*}^{(3)}(0) = 3! \cdot \dfrac{1}{6\sqrt{\lambda}} = \dfrac{1}{\sqrt{\lambda}}$