

Math 151 11-12-08.

Last time

X a r.v.

$$E(e^{tX}) = M_X'(t) \quad \text{when its defined as an integral.}$$

$$M_X'(0) = E(X)$$

$$M_X''(0) = E(X^2)$$

\vdots

$$M_X^{(k)}(0) = E(X^k).$$

We had e.g.

X Poisson parameter λ .

$$M_X(t) = e^{\lambda(e^t - 1)}$$

X exponential parameter λ .

$$M_X(t) = \frac{\lambda}{\lambda - t} \quad t < \lambda.$$

and $X \sim N(0, 1) \Rightarrow M_X(t) = e^{t^2/2}$.

e.g. if U is uniform on $[0, 1]$.

$$M_U(t) = \int_0^1 e^{tu} du = \frac{1}{t} e^{tu} \Big|_0^1 = \frac{e^t - 1}{t}.$$

$$= \frac{\sum_{k=0}^{\infty} \frac{t^k}{k!} - 1}{t} = \frac{\sum_{k=1}^{\infty} \frac{t^k}{k!}}{t} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!}.$$

$$\text{So } M_u(t) = \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!}$$

$$M_u'(0) = \frac{1}{2} \quad (k=2)$$

$$\frac{M_u''(0)}{2!} = \frac{1}{3!} \Rightarrow M_u''(0) = \frac{1}{3}. \quad \text{etc.}$$

$$\boxed{\text{Var}(u) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}}$$

On this case it is just as easy to compute moments directly.

$$E(u^k) = \int_0^1 u^k du = \frac{1}{k+1}$$

If u_1, \dots, u_n are independent and uniform on $(0,1)$.

What is $E(e^{t(u_1 + \dots + u_n)})$?

$$= E(e^{tu_1} e^{tu_2} \dots e^{tu_n})$$

$$= E(e^{tu_1}) E(e^{tu_2}) \dots E(e^{tu_n})$$

by independence.

$$= \left(\frac{e^t - 1}{t}\right)^n$$

with $U = U_1 + \dots + U_n$. U_i unif on $[0, 1]$
indep

$$V = \frac{U - n/2}{\sqrt{n/12}}$$

has mean 0, $\text{Var}(V) = 1$.

$$M_V(t) = e^{-\sqrt{12}/2 t \sqrt{n}} M_U\left(\sqrt{\frac{12}{n}} t\right)$$

recall from last time.

if $Y = aX + b$.

$$M_Y(t) = e^{bt} M_X(at)$$

$$= e^{-(\sqrt{3n})t} \left(\frac{e^{\frac{\sqrt{12}}{n} t} - 1}{\sqrt{\frac{12}{n}} t} \right)^n$$

~~$$= \left(e^{-\frac{\sqrt{3}}{n} t} \right)^n$$~~

$$= \left(e^{-\sqrt{\frac{3}{n}} t} \cdot \left(\frac{e^{2\sqrt{\frac{3}{n}} t} - 1}{2\sqrt{\frac{3}{n}} t} \right) \right)^n$$

$$= \left(\frac{e^{\sqrt{\frac{3}{n}} t} - e^{-\sqrt{\frac{3}{n}} t}}{2\sqrt{\frac{3}{n}} t} \right)^n$$

$$= \left[\frac{\left(1 + \sqrt{\frac{3}{n}} t + \frac{\left(\sqrt{\frac{3}{n}} t\right)^2}{2!} + \dots \right) - \left(1 - \sqrt{\frac{3}{n}} t + \frac{\left(\sqrt{\frac{3}{n}} t\right)^2}{2!} - \dots \right)}{2\sqrt{\frac{3}{n}} t} \right]^n$$

$$= \left(\frac{2\sqrt{\frac{3}{n}}t + 2 \frac{(\sqrt{\frac{3}{n}}t)^3}{3!} + \dots}{2\sqrt{\frac{3}{n}}t} \right)^n$$

$$= \left(1 + \frac{(\sqrt{\frac{3}{n}}t)^2}{3!} + o\left(\sqrt{\frac{3}{n}}t\right)^4 \right)^n$$

$$= \left(1 + \frac{\frac{3}{6}t^2}{n} + o\left[\left(\sqrt{\frac{3}{n}}t\right)^4\right] \right)^n$$

$$\rightarrow e^{-\frac{t^2}{2n}}$$

e.g.

if X_1, \dots, X_n are exponential w/ parameter λ and independent.

$$\boxed{E(X_1) = \frac{1}{\lambda} \quad \text{Var}(X_1) = \frac{1}{\lambda^2}}$$

$$\text{let } X = X_1 + \dots + X_n$$

$$\text{and } X^* = \frac{X - \frac{n}{\lambda}}{\sqrt{\frac{n}{\lambda^2}}} = \frac{\lambda X - \sqrt{n}}{\sqrt{n}}$$

so that $E(X^*) = 0$ and $\text{Var}(X^*) = 1$.

then

$$M_{X^*}(t) = e^{-\sqrt{n}t} M_X\left(\frac{1}{\sqrt{n}}t\right).$$

$$= e^{-\sqrt{n}t} E\left(e^{\frac{1}{\sqrt{n}}(X_1 + \dots + X_n)}\right)$$

$$= e^{-\sqrt{n}t} E\left(e^{\frac{1}{\sqrt{n}}X_1}\right) \dots E\left(e^{\frac{1}{\sqrt{n}}X_n}\right)$$

$$= e^{-\sqrt{n}t} \left(\frac{\lambda}{\lambda - \frac{1}{\sqrt{n}}t}\right)^n$$

$$= e^{-\sqrt{n}t} \left(\frac{1}{1 - \frac{t}{\sqrt{n}}}\right)^n$$

$$= e^{-\sqrt{n}t} \left(1 + \frac{t}{\sqrt{n}} + \left(\frac{t}{\sqrt{n}}\right)^2 + \left(\frac{t}{\sqrt{n}}\right)^3 + \dots\right)^n$$

$$= \left(e^{-t/\sqrt{n}} \left(1 + \frac{t}{\sqrt{n}} + \left(\frac{t}{\sqrt{n}}\right)^2 + \dots\right)\right)^n$$

$$= \left(\left(1 - \frac{t}{\sqrt{n}} + \frac{t^2}{2(\sqrt{n})^2} - \frac{t^3}{(\sqrt{n})^3} \frac{1}{3!} + \dots\right) \left(1 + \frac{t}{\sqrt{n}} + \left(\frac{t}{\sqrt{n}}\right)^2 + \dots\right)\right)^n$$

$$= \left(1 - \frac{t^2}{n} + \frac{t^2}{2n} + \frac{t^2}{n} + O\left(\frac{t^3}{n^{3/2}}\right)\right)^n$$

$$= \left(1 + \frac{t^2/2}{n} + O\left(\frac{t^3}{n^{3/2}}\right)\right)^n \rightarrow e^{t^2/2}.$$

e.g. Suppose $X_i = \begin{cases} 1 & \text{prob } p. \\ 0 & \text{prob } 1-p. \end{cases}$
and X_i are i.i.d. $i = 1, \dots, n.$

$$\text{let } X = X_1 + \dots + X_n$$

$$X^* = \frac{X - pn}{\sqrt{np(1-p)}}$$

$$\text{so that } E(X^*) = 0$$

$$\text{Var}(X^*) = 1.$$

$$M_{X_i}(t) = E(e^{tX_i}) = pe^t + (1-p).$$

$$M_X(t) = (pe^t + (1-p))^n$$

$$M_{X^*}(t) = e^{(-\sqrt{\frac{p}{1-p}} n' t)} M_X\left(\frac{t}{\sqrt{np(1-p)}}\right).$$

$$= e^{-\sqrt{\frac{pn}{1-p}} t} \left(p e^{\frac{t}{\sqrt{np(1-p)}}} + (1-p) \right)^n$$

$$= \left(e^{-\sqrt{\frac{p}{1-p}} \frac{t}{\sqrt{n}}} \left(p e^{\frac{t}{\sqrt{np(1-p)}}} + (1-p) \right) \right)^n.$$

$$\frac{1}{\sqrt{p(1-p)}} - \sqrt{\frac{p}{1-p}} = \frac{\frac{1}{\sqrt{p}} - \sqrt{p}}{\sqrt{1-p}} = \frac{\sqrt{p/p} - \sqrt{p}}{\sqrt{1-p}}$$

$$\bar{E} = \frac{\sqrt{p} \left(\frac{1}{p} - 1\right)}{\sqrt{p} \sqrt{1-p}} = \sqrt{\frac{1}{p} - 1} = \sqrt{\frac{1-p}{p}}$$

so

$$\left(p e^{\sqrt{\frac{1-p}{p}} \frac{t}{\sqrt{n}}} + (1-p) e^{-\sqrt{\frac{p}{1-p}} \frac{t}{\sqrt{n}}} \right)^n$$

$$= \left(p \left(1 + \sqrt{\frac{1-p}{p}} \frac{t}{\sqrt{n}} + \frac{\left(\sqrt{\frac{1-p}{p}} \frac{t}{\sqrt{n}} \right)^2}{2} + \frac{\left(\sqrt{\frac{1-p}{p}} \frac{t}{\sqrt{n}} \right)^3}{3!} + \dots \right) + (1-p) \left(1 - \sqrt{\frac{p}{1-p}} \frac{t}{\sqrt{n}} + \frac{\left(\sqrt{\frac{p}{1-p}} \frac{t}{\sqrt{n}} \right)^2}{2} - \frac{\left(\sqrt{\frac{p}{1-p}} \frac{t}{\sqrt{n}} \right)^3}{3!} + \dots \right) \right)^n$$

(raised to the n^{th} power)

$$= \left(1 + \frac{t^2/2}{n} + O\left(\frac{t^3}{n^{3/2}}\right) \right)^n \rightarrow e^{t^2/2}$$

Facts:

• If the moment generating function exists for t in an open interval containing 0 then it uniquely determines the probability distribution.

• Let F_n be a sequence of c.d.f. with corresponding m.g.f.'s M_n

Let F be a c.d.f w/ m.g.f M .

If $M_n(t) \rightarrow M(t) \quad \forall t$ in an open interval containing zero then $F_n(x) \rightarrow F(x)$ at all points where F is continuous.

What do these two facts say about the previous examples? -