

Math 151 10/6/08

Continuous Random Variables

X is a cont. r.v. (real valued)
if $\exists f_X(x) \geq 0$ defined on \mathbb{R}

s.t.,

$$P(X \in B) = \int_B f_X(x) dx$$

for "all" subsets $B \subset \mathbb{R}$.

e.g.

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

f_X is called the probability
density function of X . (PDF).

Notice that

$$P(X=a) = \int_a^a f_X(x) dx = 0.$$

for any $a \in \mathbb{R}$.

Remark:

We must have

$$\int_{-\infty}^{\infty} f_X(x) dx = 1. \quad \text{and } f(x) \geq 0 \\ \forall x.$$

A useful heuristic interpretation of $f_X(x)$ is given by.

$$P[X \in [x, x+\delta]] = \int_x^{x+\delta} f_X(x) \approx f_X(x) \cdot \delta.$$

so that $f_X(x)$ gives "probability per unit length".

e.g.

The uniform distribution on $[0, 1]$.

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else.} \end{cases}$$

$\text{Prob}\{a < X < b\} = b - a$ for any $0 < a < b < 1$.

e.g.

uniform distribution on $[x_0, y_0]$.

$$f_X(x) = \begin{cases} \frac{1}{y_0 - x_0} & x_0 \leq x \leq y_0 \\ 0 & \text{else.} \end{cases}$$

$$\text{Prob } \{a < X < b\} = \frac{b-a}{y_0-x_0}$$

for any $x_0 < a < b < y_0$.

In analogy with the discrete case,
we put

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

assuming the integral converges.

if g is any function, then as in
the discrete case, we have:

$$(*) \quad E[g(x)] = \int_{-A}^A g(x) f_X(x) dx.$$

Also

$$\begin{aligned} \text{Var}(X) &= E((X - E[X])^2) \\ &= \int (X - E(X))^2 f_X(x) dx \\ &= E[X^2] - E[X]^2. \end{aligned}$$

and if $Y = aX + b$ then

$$E[Y] = aE[X] + b.$$

$$\text{Var}[Y] = a^2 \text{Var}[X].$$

e.g. $X \sim \text{unif}[0, 1]$.

$$E[X] = \int_0^1 x dx = 1/2. \quad \left[\int_a^b x dx = \frac{b+a}{2} \right]$$

$$\text{Var}[X] = \int_0^1 x^2 dx - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = 1/12.$$

$$\left(\int_a^b x^2 dx - \left(\frac{b+a}{2}\right)^2 \right) = \frac{1}{3} \frac{b^3 - a^3}{b-a} - \left(\frac{b+a}{2}\right)^2$$
$$= \frac{a^2 + ab + b^2}{3} - \frac{b^2 + 2ab + a^2}{4}$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3b^2 - 6ab - 3a^2}{12}$$

$$= \frac{a^2 + b^2 - 2ab}{12}$$

$$= \frac{(b-a)^2}{12}$$

e.g. The exponential random variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else.} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = 1.$$

$$\text{Prob } \{X \geq a\} = \int_a^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_a^{\infty} = e^{-\lambda a}.$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$= -\lambda \frac{d}{d\lambda} \left(\int_0^{\infty} e^{-\lambda x} dx \right) = -\lambda \left[\frac{d}{d\lambda} \left(\frac{1}{\lambda} \right) \right]$$

$$= -\lambda \frac{d}{d\lambda} \left[-\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty} \right]$$

$$= -\lambda \frac{d}{d\lambda} \left[\frac{1}{\lambda} \right] = +\frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

$$E[X^2] = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda \frac{d^2}{d\lambda^2} \left[\int_0^{\infty} e^{-\lambda x} dx \right] = \lambda \frac{d^2}{d\lambda^2} \left[\frac{1}{\lambda} \right] = \frac{2}{\lambda^2}$$

$$E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

Cumulative distribution functions (C.D.F)

Given a r.v. (of any type) X , the c.d.f of X is

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} P_X(k) \quad \text{disc.}$$
$$= \int_{-\infty}^x f_X(t) dt. \quad \text{con.}$$

In the continuous case

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt.$$

$$\text{so } F_X'(x) = f_X(x).$$

e.g. Let X be a Poisson variable w/ parameter λ .

What is the probability that no events occur up to time t .

events in $[0, t]$ is Poisson
with parameter λt

$$\text{Prob } \{ \text{no events} \} = e^{-\lambda t} \\ \text{in } [0, t].$$

$$\therefore \text{Prob (at least one event in } [0, t]) \\ = 1 - e^{-\lambda t}$$

Let $Y =$ time until 1st event

$$\text{Prob } \{ Y \leq t \} = 1 - e^{-\lambda t}$$

$$\therefore f_Y(t) = \lambda e^{-\lambda t}.$$

(Y is exponential w/ parameter λ).

e.g.

Let X be geometric w/ parameter p .

$$F_X(n) = \sum_{k=1}^n p(1-p)^{k-1} = p \sum_{j=0}^{n-1} (1-p)^j \\ = p \frac{1 - (1-p)^n}{1 - (1-p)} = 1 - (1-p)^n.$$

Normal Random Variables.

$$X \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Trick:

$$I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2} dx \int_{-\infty}^{\infty} e^{-y^2/2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dx dy$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2} d\theta (r dr)$$

$$= 2\pi \int_0^{\infty} e^{-r^2/2} r dr.$$

$$u = r^2/2 \quad du = r dr.$$

$$= 2\pi \int_0^{\infty} e^{-u} du = 2\pi (-e^{-u}/0^{\infty}) = 2\pi.$$

So $I = \sqrt{2\pi}.$

$$\therefore \text{for } \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\text{put } u = \left(\frac{x-\mu}{\sigma}\right) \quad du = \frac{1}{\sigma} dx.$$

$$= \frac{\sigma}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-u^2/2} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du = 1.$$

$$E[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} (x) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\text{but } \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 0.$$

odd-symmetric about $x=\mu$.

$$\text{so } E(X) = \mu.$$

$$\text{Var}(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$\frac{d}{d\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{+2(x-\mu)^2}{2\sigma^3} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{d}{d\sigma} [\sqrt{2\pi}\sigma] = \sqrt{2\pi}.$$

$$\text{so } \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \sqrt{2\pi} \sigma^3$$

$$\text{and } \therefore \text{Var}(x) = \sigma^2$$