

Math 151 9/29

Homework questions (soln's to 2.2.74, 2.2.72 appended).

→ Joint PMF's.

$X, Y$  are discrete random variables

The joint PMF of  $(X, Y)$  is

$$\begin{aligned} P_{X,Y}(x,y) &= \text{Prob} \{ X=x, Y=y \} \\ &= \text{Prob} \left( \{ X=x \} \cap \{ Y=y \} \right). \end{aligned}$$

If  $A$  is any set of pairs  $(x,y)$

then

$$P((X,Y) \in A) = \sum_{(x,y) \in A} P_{X,Y}(x,y)$$

e.g.

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

since

$$P(X=x) = \sum_y P(X=x, Y=y)$$

likewise

$$P_Y(y) = \sum_x P_{X,Y}(x,y).$$

If  $Z = g(X, Y)$  then

$$\text{Prob}\{g(X, Y) = z\} = P_Z(z) = \sum_{\{(x, y) | g(x, y) = z\}} P_{X, Y}(x, y).$$

and

$$E(g(X, Y)) = \sum_x \sum_y g(x, y) P_{X, Y}(x, y).$$

Pf:  $P(Z = z) = \sum_{(x, y) : g(x, y) = z} P_{X, Y}(x, y)$

$$\text{so, } \sum_z z P_Z(z) = \sum_z z \sum_{\{(x, y) : g(x, y) = z\}} P_{X, Y}(x, y)$$

$$= \sum_z \sum_{\{(x, y) : g(x, y) = z\}} z P_{X, Y}(x, y)$$

$$= \sum_z \sum_{\{(x, y) : g(x, y) = z\}} g(x, y) P_{X, Y}(x, y)$$

$$= \sum_{(x, y)} g(x, y) P_{X, Y}(x, y).$$

e.g.

$$E[aX + bY + c]$$

$$= \sum_x \sum_y (ax + by + c) P_{X,Y}(x,y)$$

$$= a \sum_x \sum_y x P_{X,Y}(x,y) + b \sum_x \sum_y y P_{X,Y}(x,y) + c \sum_x \sum_y P_{X,Y}(x,y).$$

$$= a \sum_x x \sum_y P_{X,Y}(x,y) + b \sum_y y \sum_x P_{X,Y}(x,y) + c.$$

$$= a \sum_x x P_X(x) + b \sum_y y P_Y(y) + c.$$

$$= a E(X) + b E(Y) + c.$$

e.g. Mean of a Binomial

if  $X \sim b(n, k, p)$

(i.e.  $\text{Prob} \{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$ ).

$$X = X_1 + X_2 + \dots + X_n$$

where each  $X_i$  is Bernoulli

with  $\text{prob} \{X_i=1\} = p$   $\text{prob} \{X_i=0\} = 1-p$ .

$$E(X) = \sum_{i=1}^n E(X_i) = np.$$

e.g. Mean of a "negative binomial"

Probability that  $r^{\text{th}}$  success

occurs on the  $k^{\text{th}}$  trial

$$\text{Prob} \{X=k\} = \binom{k-1}{r-1} p^{r-1} \cdot (1-p)^{k-r} \cdot p$$

$$X = X_1 + X_2 + \dots + X_r$$

$X_i$  is geometric w/ prob  $p$  of success

$$\text{so } E(X) = \frac{r}{p}.$$

2.2.72. Mean and Variance of a hypergeometric random variable  $X$ .

$N, r, n$  fixed

$$P(X=x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}$$

With  $m = \min(r, n)$ ,

$$E(X) = \sum_{x=0}^m x \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \sum_{x=1}^m x \binom{r}{x} \binom{N-r}{n-x}$$

Note that  $\binom{x}{r} \binom{r}{x} = \binom{r-1}{x-1}$  to get

$$E(X) = \frac{r}{\binom{N}{n}} \sum_{x=1}^m \binom{r-1}{x-1} \binom{N-r}{n-x}$$

We have the combinatorial identity

$$\binom{N}{n} = \sum_{x=0}^n \binom{r}{x} \binom{N-r}{n-x} = \sum_{x=0}^{\min(r, n)} \binom{r}{x} \binom{N-r}{n-x}.$$

Since  $\sum_{x=1}^m \binom{r-1}{x-1} \binom{N-r}{n-x} = \sum_{k=0}^{m-1} \binom{r-1}{k} \binom{N-1-(r-1)}{(n-1)-k}$

$$= \binom{N-1}{n-1} \quad (\text{by the identity}),$$

we have

$$E(X) = \frac{r}{\binom{N}{n}} \binom{N-1}{n-1} = \frac{r}{N} \cdot n$$

2.2.72 cont.

$$\begin{aligned}\text{Var}(X) &= E(X^2) - E(X)^2 \\ &= E(X(X-1)) + E(X) - E(X)^2 \\ &= E(X(X-1)) + \frac{rn}{N} - \left(\frac{rn}{N}\right)^2\end{aligned}$$

$$\begin{aligned}E(X(X-1)) &= \sum_{x=0}^m x(x-1) \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \sum_{x=2}^m x(x-1) \binom{r}{x} \binom{N-r}{n-x} \\ &= \frac{r}{\binom{N}{n}} \sum_{x=2}^m (x-1) \binom{r-1}{x-1} \binom{N-r}{n-x} \\ &= \frac{r(r-1)}{\binom{N}{n}} \sum_{x=2}^m \binom{r-2}{x-2} \binom{N-r}{n-x} \\ &= \frac{r(r-1)}{\binom{N}{n}} \sum_{k=0}^{m-2} \binom{r-2}{k} \binom{N-2-(r-2)}{(n-2)-k} \\ &= \frac{r(r-1)}{\binom{N}{n}} \binom{N-2}{n-2} = \frac{r(r-1) \cdot n \cdot (n-1)}{N(N-1)}\end{aligned}$$

So

$$\begin{aligned}\text{Var}(X) &= \frac{r(r-1) \cdot n \cdot (n-1)}{N(N-1)} + \frac{rn}{N} \left(1 - \frac{rn}{N}\right) \\ &= \frac{rn}{N} \left(1 - \frac{rn}{N} + \frac{(r-1)(n-1)}{N-1}\right)\end{aligned}$$

etc..

## 2.2.74. Mean and Variance of a negative binomial

$$p, r \text{ fixed} \quad P_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$E(X) = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

We have the following Maclaurin Series

$$\frac{1}{(1-q)^r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} q^{x-r}$$

$\frac{d}{dq}$   $\downarrow$

$$\frac{r}{(1-q)^{r+1}} = \sum_{x=r+1}^{\infty} (x-r) \binom{x-1}{r-1} q^{x-r-1}$$

So we compute

$$E(X-r) = \sum_{x=r}^{\infty} (x-r) \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$= p^r (1-p) \sum_{x=r+1}^{\infty} (x-r) \binom{x-1}{r-1} (1-p)^{x-r-1}$$

$$= p^r (1-p) \cdot \frac{r}{p^{r+1}} \quad (\text{using the 2nd series with } q=1-p)$$

$$= \frac{(1-p)r}{p}$$

and we have

$$E(X) = \frac{(1-p)r}{p} + r = \frac{r}{p}$$

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differentiating our second series, we have

$$\frac{r(r+1)}{(1-q)^{r+2}} = \sum_{x=r+2}^{\infty} (x-r)(x-r-1) \binom{x-1}{r-1} q^{x-r-2}$$

So we can compute

$$E((X-r)(X-r-1)) = \sum_{x=r+2}^{\infty} (x-r)(x-r-1) \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$= p^r (1-p)^2 \sum_{x=r+2}^{\infty} (x-r)(x-r-1) \binom{x-1}{r-1} (1-p)^{x-r-2}$$

$$= p^r (1-p)^2 \frac{r(r+1)}{p^{r+2}} \quad \left( \text{using the last series with } q=1-p \right)$$

$$= \frac{(1-p)^2}{p^2} r(r+1).$$

$$\text{Now } (X-r)(X-r-1) = X^2 - (2r+1)X + r(r-1)$$

$$\text{So } E(X^2) = \frac{(1-p)^2}{p^2} r(r+1) + (2r+1) \left( \frac{r}{p} \right) - r(r-1)$$

$$\text{and } \text{Var}(X) = \frac{(1-p)^2}{p^2} r(r+1) + (2r+1) \left( \frac{r}{p} \right) - r(r-1) - \left( \frac{r}{p} \right)^2.$$

which simplifies to

$$\frac{r \cdot (1-p)}{p^2}.$$