

Math 151 9/24.

Mean of a discrete R.V. X .

$$E[X] = \sum_x x P_X(x).$$

a weighted average of possible values of X .

Variance of a discrete R.V X

$$E((X - E(X))^2)$$

$$= E(X^2 - 2X E(X) + E(X)^2)$$

= (as we shall see next time).

$$E(X^2) - 2 E(X) E(X) + E(X)^2$$

$$= E(X^2) - (E(X))^2.$$

~~More on those next class~~.

$$\text{Var}(X) = E((X - E(X))^2)$$

$$= E(X^2) - (E(X))^2.$$

The variance is an algebraically convenient measure of how spread out the values of X typically are about $E(X)$.

Expected value Rule for functions
of random variables:

If X is a R.V with p.m.f $P_X(x)$
and $Y = g(X)$ for a given function
 g defined on the range of X then

$$E(g(X)) = \sum_y y P_Y(y) = \sum_x g(x) P_X(x).$$

Pf:

$$\text{Prob}\{Y=y\} = P_Y(y) = \sum_{\{x: g(x)=y\}} P_X(x)$$

$$\begin{aligned} \text{so } E(g(X)) &= \sum_y y P_Y(y) \\ &= \sum_y y \sum_{\{x: g(x)=y\}} P_X(x) \\ &= \sum_y \sum_{\{x: g(x)=y\}} y P_X(x) = \sum_y \sum_{\{x: g(x)=y\}} g(x) P_X(x) \\ &= \sum_x g(x) P_X(x). \end{aligned}$$

e.g.

$$E((X - E(X))^2)$$

$$= \sum_x (x - E(X))^2 P_X(x)$$

$$= \sum_x (x^2 - 2x E(X) + E(X)^2) P_X(x)$$

$$= \sum_x x^2 P_X(x) - 2 E(X) \sum_x x P_X(x) + E(X)^2 \sum_x P_X(x)$$

$$= E(X^2) - 2(E(X))^2 + E(X)^2$$

$$= E(X^2) - E(X)^2.$$

e.g.

$$Y = aX + b \quad , \text{ } a, b \text{ constants} .$$

$$E[Y] = \sum_x (ax + b) P_X(x)$$

$$= a \sum_x x P_X(x) + b \sum_x P_X(x)$$

$$= a E(X) + b.$$

e.g. cont.

$$\begin{aligned}\text{Var}(Y) &= E(Y^2) - E(Y)^2 \\&= \sum_x (ax+b)^2 P_X(x) - (aE(X)+b)^2 \\&= \sum_x (a^2x^2 + 2axb + b^2) P_X(x) \\&\quad - a^2 E(X)^2 - 2ab E(X) \\&\quad - b^2. \\&= a^2 E(X^2) + 2ab E(X) + b^2 - a^2 E(X)^2 - 2ab E(X) - b^2 \\&= a^2(E(X^2) - E(X)^2) = a^2 \text{Var}(X).\end{aligned}$$

The last two examples should be remembered.

$$\bullet E(ax+b) = aE(x) + b.$$

$$\bullet \text{Var}(ax+b) = a^2 \text{Var}(x)$$

e.g. If X is Bernoulli with

$$P_X(1) = p$$

$$P_X(0) = 1-p.$$

$$E(X) = p \cdot 1 + (1-p) \cdot 0 = p.$$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$= p - p^2 = p(1-p)$$

e.g. X = face value of a regular dice roll

$$E(X) = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) \\ = 3.5.$$

$$\text{Var}(X) = \frac{1}{6} (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - (3.5)^2$$

$$= 35/12.$$

e.g. X = # of rolls of a dice until 1st 6 appears.

$$\text{Prob } \{X=k\} = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$$

$$E(X) = \frac{1}{6} \sum_{k=1}^{\infty} k \cdot \left(\frac{5}{6}\right)^{k-1}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad -1 < x < 1.$$

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} \quad -1 < x < 1.$$

$$\text{so } E(X) = \frac{1}{6} \cdot \frac{1}{(1-\frac{5}{6})^2} = 6$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - 36.$$

$$E(X^2) = \frac{1}{6} \sum_{k=1}^{\infty} k^2 \left(\frac{5}{6}\right)^{k-1}$$

Notice that

$$\frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k(k-1) x^{k-2}$$

so that

$$\begin{aligned} E(X(X-1)) &= \frac{1}{6} \sum_{k=1}^{\infty} k(k-1) \left(\frac{5}{6}\right)^{k-1} \\ &= \frac{1}{6} \cdot \frac{5}{6} \sum_{k=2}^{\infty} k(k-1) \left(\frac{5}{6}\right)^{k-2} \\ &= \frac{1}{6} \cdot \frac{5}{6} - \frac{2}{(1-5/6)^3} = 60 \end{aligned}$$

We have $E(X) = 6$ and

$$\begin{aligned} E(X^2) &= E(X(X-1) + X) = \frac{1}{6} \sum_{k=1}^{\infty} (k(k-1) + k) \left(\frac{5}{6}\right)^{k-1} \\ &= 66 \end{aligned}$$

$$\therefore \text{Var}(X) = 66 - 36 = 30.$$

In general, if X is geometric
(with prob p of success)

$$E(X) = \frac{1}{p}$$

$$\begin{aligned} \text{Var}(X) &= p(1-p) \cdot \frac{2}{p^3} + \frac{1}{p} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2}. \end{aligned}$$

$$\text{e.g. } \text{Prob}\{X = k\} = e^{-\lambda} \frac{\lambda^k}{k!} \quad k=0, 1, 2, \dots$$

i.e. X is Poisson w/ parameter $\lambda > 0$.

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \\ &= \lambda e^{-\lambda} e^{\lambda} = \lambda. \end{aligned}$$

$$\text{Var}(X) = E(X^2) - \lambda^2$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!}$$

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$\frac{d}{d\lambda} e^{\lambda} = e^{\lambda} = \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{k!}$$

$$\frac{d^2}{d\lambda^2} e^{\lambda} = e^{\lambda} = \sum_{k=2}^{\infty} k(k-1) \frac{\lambda^{k-2}}{(k-2)!}$$

$$k^2 = k(k-1) + k$$

$$\text{So } E(X^2) = e^{-\lambda} \sum_{k=0}^{\infty} (k(k-1) + k) \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \left[\sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \right]$$

$$= e^{-\lambda} \left[\lambda^2 \sum_{k=2}^{\infty} k(k-1) \frac{\lambda^{k-2}}{k!} + \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{k!} \right]$$

$$= e^{-\lambda} \left[\lambda^2 e^\lambda + \lambda e^\lambda \right] = \lambda^2 + \lambda.$$

and

$$E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

e.g.

$$X \sim b(n, p).$$

$$E(X) = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} = np.$$

$$\text{Pf. } \Leftrightarrow \sum_{k=1}^n \left(\frac{k}{n} \right) \binom{n}{k} p^{k-1} (1-p)^{n-k} = 1.$$

$$\Leftrightarrow \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} = 1.$$

$$\Leftrightarrow \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} = 1. \quad \checkmark.$$

Later, we will see that an

$$X \sim b(n, k, p)$$

is a sum of independent

Bernoulli R.V's.

$$X = X_1 + \dots + X_n$$

$$\text{where } X_j = \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } 1-p \end{cases}$$

We will have

$$\begin{aligned} E(X) &= E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n) \\ &= np. \end{aligned}$$

(Linearity of expectations)

and

$$\text{Var}(X) = (\text{by } \underline{\text{independence}})$$

$$\sum_{j=1}^n \text{Var}(X_j) = n \cdot p(1-p).$$