

Math 151 9/15/08

Counting Principles
(continued from last time).

From (a_1, \dots, a_m)
 (b_1, \dots, b_n)

how many pairs (a_i, b_j)
can be formed?

\boxed{mn}

(place them in an array)

From (a_1, \dots, a_{m_1})
 (b_1, \dots, b_{n_1})
:
 (x_1, \dots, x_{n_r})

}

, groups.

how many r-tuples

$(a_{i_1}, b_{i_2}, \dots, x_{i_r})$

can be formed?

$\boxed{n_1 n_2 n_3 \dots n_r}$

(by induction and the previous
result)

From a set of n objects, in
how many ways can k things
be chosen when replacement is allowed?

$$\boxed{n^k}$$

When replacement is not allowed?

$$\boxed{n \cdot (n-1) \cdots (n-(k-1))}$$

(There are both applications of the 2nd
counting principle).

In how many distinct ways can
 n objects be placed in order?

$$\boxed{n \cdot (n-1) \cdots \cdot 2 \cdot 1 = n!}$$

If k people are chosen randomly
what is the probability that they
all have different Birthdays?

How large must k be so that this is $< \frac{1}{2}$?

Assume 365 equally likely Birthdays.

$$P_n = \frac{(365)(364)\dots(365-(k-1))}{(365)^k}$$

$$= \left(1 - \frac{1}{365}\right) \left(1 - \frac{2}{365}\right) \dots \left(1 - \frac{k-1}{365}\right)$$

$$\log P_n = \sum_{n=1}^{k-1} \log \left(1 - \frac{n}{365}\right)$$

The smallest k for which

$$\sum_{n=1}^{k-1} \log \left(1 - \frac{n}{365}\right) < \log \frac{1}{2} \approx k=23.$$

Qn how many ways can a subset of size k be chosen from a set of n objects?

(Two sets are the same if they contain the same elements).

Call the answer $\binom{n}{k}$, then

$$n \cdot (n-1) \cdots (n-(k-1)) = \binom{n}{k} \cdot k!$$

(by our 1st counting principle) so

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k!} = \frac{n!}{(n-k)!k!}$$

The numbers $\binom{n}{k}$ are called

Binomial Coefficients because

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Derivation

$$(a+b)^n = \underbrace{(a+b)(a+b) \dots \dots (a+b)}_{n \text{ factors}}$$

Formation of a monomial $a^{n-k} b^k$ involves selecting k of the parentheses from which to choose a "b" and taking an "a" from the remaining $(n-k)$ parentheses. This can be done in $\binom{n}{k}$ ways.

$$\text{Set } \binom{n}{-1} = \binom{n}{n+1} = 0.$$

$$\text{We have: } (a+b) \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

$$= \sum_{k=0}^n \binom{n}{k} (a^{n-k+1} b^k + a^{n-k} b^{k+1})$$

$$= \sum_{k=0}^n \binom{n}{k} a^{n-k+1} b^k + \sum_{j=1}^{n+1} \binom{n}{j-1} a^{n-j+1} b^j \quad (k+1 = j)$$

$$\text{Now we } \binom{n}{-1} = \binom{n}{n+1} = 0$$

to see that

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^k$$

so that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

This is the relation usually represented by Pascal's triangle

$n=0$	1
$n=1$	(1)
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$(n \leq 5)$	$\underbrace{1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1}_{k}$

Putting $a=b=1$ we get

$$2^n = \sum_{k=0}^n \binom{n}{k} \quad \text{and this}$$

formally corresponds to counting the

of subsets of a set of size n
 in 2 different ways.

Putting $a=1$ $b=x$

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

and differentiating we get.

$$n(1+x)^{n-1} = \sum_{k=1}^n \binom{n}{k} k x^{k-1}$$

and setting $x=1$ we have

$$n \cdot 2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

and this counts the number of clubs with a leader from a group of size n .

(in two different ways).

What is the probability of tossing exactly 50 heads in 100 tosses of a fair coin?

$$\binom{100}{50} \cdot \frac{1}{2^{100}} \approx \quad \checkmark$$

If the probability of a head is p ($0 < p < 1$) what is the probability of exactly k heads in n independent tosses?

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Notice that

$$1 = (p + (1-p))^n = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}.$$

5 cards are dealt from a shuffled deck.

What is the probability that there is no pair?

$$\frac{\binom{13}{5} \cdot 4^5}{\binom{52}{5}} \approx .5071.$$

Now suppose that

$$n = r_1 + r_2 + \dots + r_k \quad r_i \geq 0 \text{ integers.}$$

In how many ways can n elements be divided into k subsets so that the 1st contains r_1 elements, the 2nd r_2 elements etc.

$$\frac{n!}{r_1! r_2! r_3! \dots r_k!}$$

$$= \binom{n}{r_1} \binom{n-r_1}{r_2} \binom{n-r_1-r_2}{r_3} \dots \binom{n-(r_1+\dots+r_{n-1})}{r_{n-1}}$$

$$= \frac{n!}{r_1!(n-r_1)!} \cdot \frac{(n-r_1)!}{r_2!(n-r_1-r_2)!} \cdot \frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!} \dots$$

If we throw 12 dice, what is the probability of 6 distinct doubles?

$$\left(\frac{\frac{12!}{2!2!2!2!2!2!}}{6^{12}} \right) \approx 0.003438\dots$$

Divide a shuffled deck into 4 hands of size 13.

What is the probability that each contains an ace?

$$\frac{52!}{13!13!13!13!} \text{ total ways to deal.}$$

$4!$ ways to distribute the aces.

$\frac{48!}{12! \cdot 12! \cdot 12! \cdot 12!}$ ways to distribute the remaining cards.

$$\left(\frac{\frac{4! \cdot 48!}{12! \cdot 12! \cdot 12! \cdot 12!}}{\frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!}} \right) = \frac{24 \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = .105\dots$$