Math 151 9/8/08

Probability Models

consist of a Sample Space

and a Probability Law.

The sample space is the set of possible outcomes of an "experiment".

e.g. (experiments).

throw a die

toss 3 coins.

? let the computer generate a random number

? call a person on the phone to ask what party they belong to?

? sample the atmospheric background noise for 1 second.

? Record the time it takes a light bulb to wear out.

? Record the number of flying bomb hits in a square block during the WW II, in London.

We denote the sample space by Ω.

A probability law assigns to any

<u>set of possible outcomes</u> A
(Event)

a number $0 \leq P(A) \leq 1$.

The numbers $P(A)$ should reflect our knowledge/belief about the likelihood of A.

We require the numbers $P(A)$ to satisfy

1. $P(A) \geq 0 \quad \forall A \subset \Omega$

2. $A, B \subset \Omega \quad A \cap B = \emptyset \Rightarrow$

$\Rightarrow P(A \cup B) = P(A) + P(B)$.

3. If $A_1, A_2, \ldots A_n, \ldots \subset \Omega$ pairwise

and $A_m \cap A_n = \emptyset \quad \forall m \neq n$.

then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$.

4. $P(\Omega) = 1$.

e.g. experiment "roll two regular dice".

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

$$P\{2\} = 1/36 = P\{12\}$$

$$P\{3\} = 2/36 = P(\{11\})$$

$$P\{4\} = 3/36 = P\{10\}$$

$$P\{5\} = 4/36 = P\{9\}$$

$$P\{6\} = 5/36 = P\{8\}$$

$$P\{7\} = 6/36$$

ii) Same experiment.

$$\Omega = \{\{i, j\} : 1 \le i \le 6,\ 1 \le j \le 6\}.$$

$$P[\{i, j\}] = 1/36 \quad \text{for each } 1 \le i \le 6,\ 1 \le j \le 6.$$

Both of the previous examples are called Discrete Probability laws since the law is specified for each element of the sample space.
The examples show that we have some freedom/discretion in choosing a sample space.
The second example is a <u>discrete uniform law</u>. In such a case, the probability of an event A is

$$\text{Prob}\{A\} = \frac{\#(A)}{\#(\Omega)}.$$

<u>Continuous Models</u> If we try to model the choice of a random number R between (say) 0 and 1.

with all choices being equally likely, we run into a problem.

If $\epsilon = P\{R = x\} > 0$ then for distinct points $\{x_1, \ldots, x_N\}$,

$$P\{R \in \{x_1, \ldots, x_N\}\} = N\epsilon > 1$$

if $N > 1/\epsilon$.

Instead, we model the situation by setting $\text{Prob}\{R \in (a,b)\} = (b-a)$

for $0 \le a < b \le 1$.

e.g. <u>The reluctant duelers</u>

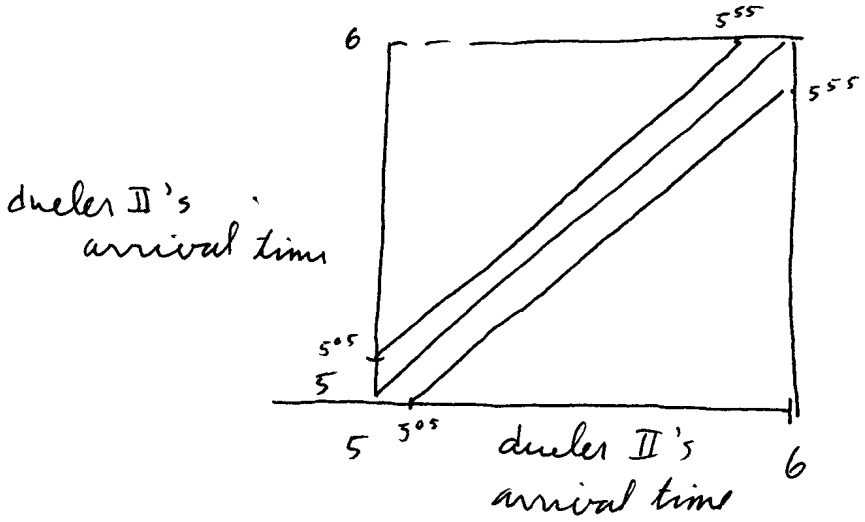
"Meet with pistols in the park between 5 and 6 p.m.!"

Yet common courtesy allows all matters of honor to be

satisfied if one's opponent does not appear within 5 minutes of one's arrival at the park. What is the probability of a gunfight?

We assume all arrival times are equally likely for each player.

We take the sample space to be



$\Omega = \{$ (ordered) pairs of arrival times $\}$

in units of hours to keep the right normalization.

The law is $P\{I \in (a,b), II \in (c,d)\} = (b-a)(d-c)$.

$$\text{Prob}\{\text{gunfight}\} = 1 - \left(\frac{11}{12}\right)^2$$

$$= \frac{23}{144}.$$

Properties of probability laws.

i) $A \subseteq B \Rightarrow P(A) \leq P(B)$

since $B = A \cup B|A = A \cup (B \cap A^c)$

and $\therefore P(B) = P(A) + P(B|A)$.

ii) $\text{Prob}\{A \cup B\} = \text{Prob}\left(A|B \cup (A \cap B) \cup B|A\right)$

$= \text{Prob}\{A|B\} + \text{Prob}\{A \cap B\} + \text{Prob}\{B|A\}$

$= \text{Prob}\{A\} - \text{Prob}\{A \cap B\} + \text{Prob}\{A \cap B\}$

$+ \text{Prob}\{B\} - \text{Prob}\{A \cap B$

$= \text{Prob}(A) + \text{Prob}(B) - \text{Prob}(A \cap B)$.

$\therefore$ (also

iii) $\text{Prob}\{A \cup B\} \leq \text{Prob}\{A\} + \text{Prob}\{B.\}$

Inclusion-Exclusion

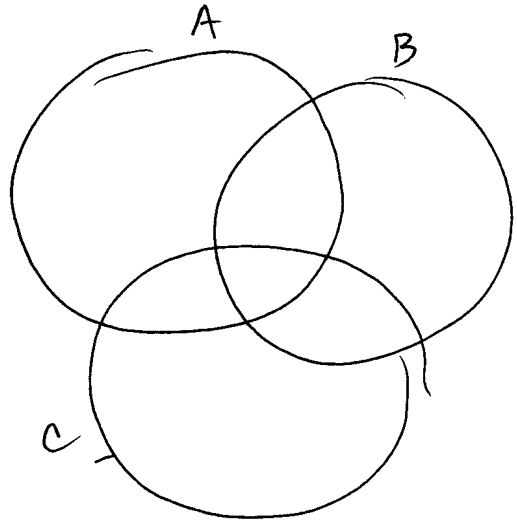
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C))$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - \left(P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)\right)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



Claim:

If $A_1, \dots A_n$ are events

put $S_1 = \{ i \mid 1 \le i \le n \}$

$S_2 = \{ (i_1, i_2) \mid 1 \le i_1 < i_2 \le n \}$

$\vdots$

$S_m = \{ (i_1, \dots, i_m) \mid 1 \le i_1 < i_2 < \dots < i_m \le n \}$

then

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{i \in S_1} P(A_i) - \sum_{(i_1, i_2) \in S_2} P(A_{i_1} \cap A_{i_2})$$

$$+ \sum_{S_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \dots + (-1)^{n-1} P\left(\bigcap_{k=1}^{n} A_k\right).$$

Pf: (By induction)

We know the cases $n=2$, $n=3$.

Suppose we know the formula holds for $n \le N$ and consider

$$P\left(\bigcup_{k=1}^{N+1} A_k\right) = P\left(\bigcup_{k=1}^{N+1} A_k \cup A_{N+1}\right).$$

$$= P\left(\bigcup_{k=1}^{N} A_k\right) + P(A_{N+1}) - P\left(\left(\bigcup_{k=1}^{N} A_k\right) \cap A_{N+1}\right)$$

$$= \sum_{i \in S_1(N)} P(A_i) - \sum_{(i_1,i_2) \in S_2(N)} P(A_{i_1} \cap A_{i_2}) + \cdots (-1)^{N-1} P\left(\bigcap_{k=1}^{N} A_k\right)$$

$$+ P(A_{N+1}) - P\left(\bigcup_{k=1}^{N} (A_k \cap A_{N+1})\right)$$

$$= \sum_{i \in S_1(N+1)} P(A_i) - \sum_{(i_1,i_2) \in S_2(N)} P(A_{i_1} \cap A_{i_2}) + \cdots + (-1)^N P\left(\bigcap_{k=1}^{N} A_k\right)$$

$$- \left(\sum_{i \in S_1(N)} A_i \cap A_{N+1} - \sum_{(i_1,i_2) \in S_2(N)} (A_{i_1} \cap A_{i_2} \cap A_{N+1}) \right.$$

$$\left. + \sum_{(i_1,i_2,i_3) \in S_3(N)} (A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{N+1}) - \cdots \right.$$

$$\left. (-1)^{N-1} P(A_1 \cap \cdots \cap A_N \cap A_{N+1}) \right)$$

and this gives the formula for the case $N+1$.

Conditional Probabilities

$P(A|B)$ = "Probability of A given B".

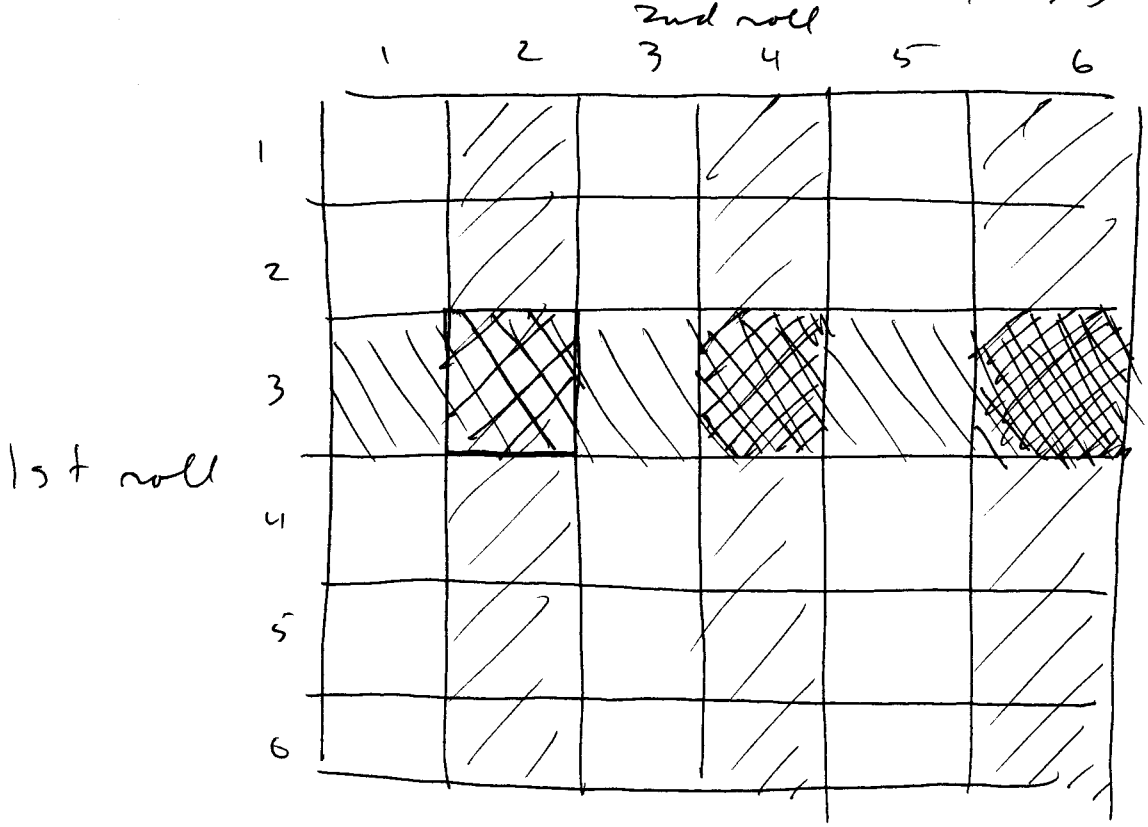
Consider first the case when A and B are <u>independent</u>.

Then $P(A|B) = P(A)$.

e.g. 2 rolls of a die

A = 2nd roll is even

B = 1st roll is a 3.



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/36}{6/36} = 1/2 = P(A)$$

In this case $P(A \cap B) = P(A)P(B)$

and we can take this as a definition of the independence of the events A and B.

In general $P(A \cap B) \neq P(A)P(B)$

and then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

reflects our updated information about A given that B has occurred.

e.g. (a very simple one).

On a roll of 3 dice

$A = \{$ the sum is odd $\}$

$B = \{$ at least one die shows odd $\}$

$P(B) = 7/8$ but $P(B|A) = 1$ since $A \subset B$.

e.g.

people are chosen at random from a population and the following attributes are recorded

S = smoker
NS = non-smoker
D = regular users of alcohol
ND = occasional users of alcohol or teetotallers.

$$Prob\{D/S\} = Prob\{D\} \; ?$$

In theory, how would we find the "exact" value of

$$Prob\{B/A\} \; ?$$

Total Probability and Bayes Rule

Bayes Rule

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \qquad \text{(Bayes Rule)}$$

If $A_1, \ldots, A_n$ are disjoint events and $\Omega = A_1 \cup \ldots \cup A_n$ then for any $B \subset \Omega$.

$$B = (A_1 \cap B) \cup \ldots \cup (A_n \cap B)$$

and the union is disjoint

So $$P(B) = P(A_1 \cap B) + \ldots + P(A_n \cap B)$$

$$= P(B|A_1)P(A_1) + \ldots P(B|A_n)P(A_n)$$

This is sometimes (confusingly) called the law of total probability. Combining it with Bayes Rule we have...

$$P(A_i|B) = \boxed{\frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \ldots + P(B|A_n)P(A_n)}}$$

and this is also sometimes called "Bayes Rule".

In many applications, computing the R.H.S. (box) is more straightforward.

e.g. <u>False positives</u>

A test for a bad disease has a negligible rate of false negatives (i.e. if someone has the disease, the test nearly always reveals it).

The test has a 5% rate of false positives.

(i.e. a healthy subject has a 5% chance to test positive).

The disease rate in the effected population is $1/1000$.

If a subject tests positive, what is the probability they are infected?

Let H denote the event {Healthy}

S " " " {Sick}

$+$ " " " {test positive}

$-$ " " " {test negative}.

We want to know

$$P\{S|+\} = \frac{P\{+|S\}P\{S\}}{P\{+\}}$$

$$= \frac{P\{S\}}{P\{+\}} = \frac{.001}{P(+)}$$

$$P(+) = P(+|S)P(S) + P(+|H)P(H)$$
$$= .001 + (.05)(.999)$$

$$\text{So} \quad P(S \mid +) \cong \frac{.001}{.001 + (.05)(.999)}$$

$$\cong \frac{1}{50}.$$

e.g. The Prisoner's dilemma (see problem 24 pg 58).

A = Prisoner 1 is released

B = Prisoner 1 is told 2 will be released

C = 1 is told 3 will be released.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/2} = 2/3.$$

and similarly $P(A \mid C) = 2/3$.

Further (various) examples.

e.g. The monty hall problem

A prize is behind one of three doors. [redacted]

You choose a door. Monty opens one of the other doors after making sure there is no prize behind it.
(Monty is not trying to trick you and you have no reason to think he prefers any door or chooses other than at random).
You may stay with your choice, or switch. What should you do?

Other examples

#5, #8, #40.

assigned problems on dice.

turn these in together with assignment 1 next Monday.

Schaum's

1.71, 1.72, 1.73, 1.75, 1.76, 1.78
1.79, 1.80, 1.81, 1.82.

all answers must be completely explained