

Math 151 9-3-08

The following lecture is meant as an introduction to some of the ideas we will encounter in this course. For next time (Mon. 9-8-08) please read Chapter 1 of [BT], try as many problems as you can, and hand in 1, 2, 6, 14, 49 [BT]. In remaining lectures I will follow the order of topics in [BT] and will give weekly homework assignments of about 10 problems from [BT] and [H]
(Schaum's)

Craps

A shooter rolls two regular dice

- 7 or 11 on the first roll is a win
- 2, 3 or 12 on the first roll is a loss
- 4, 5, 6, 8, 9, 10 on the first roll is called the shooter's "point".

If the first roll is a "point" then the shooter continues rolling the dice until either a 7 shows or the same "point" is rolled. A roll of 7 is a loss while rolling the point is a win.

Question: What chance does the shooter have to win?

Probabilities of mutually exclusive events should add. So we expect

$$\begin{aligned}\text{Prob}\{\text{Win}\} &= \text{Prob}\{7 \text{ on 1st roll}\} \\&\quad + \text{Prob}\{11 \text{ on 1st roll}\} \\&\quad + \text{Prob}\{\text{Win with point}=4\} \\&\quad + \text{Prob}\{\text{Win with point}=5\} \\&\quad + \text{Prob}\{\text{Win with point}=6\} \\&\quad + \text{Prob}\{\text{Win with point}=8\} \\&\quad + \text{Prob}\{\text{Win with point}=9\} \\&\quad + \text{Prob}\{\text{Win with point}=10\}\end{aligned}$$

What is

$$\text{Prob}\{7 \text{ on 1st roll}\} ?$$

When we roll 2 dice, there are 36 equally likely possible outcomes.

1st die

1

2

3

4

5

6

2nd die

{1, 2, 3, 4, 5, 6}

"

"

.

.

.

The set of 36 outcomes is called a sample space.

The assignment of equal probability ($\frac{1}{36}$) to each possible outcome is an example of a probability measure on a sample space.

Which outcomes correspond to the event {7 on 1st roll}?

In the finite sample space of 36 outcomes for one dice roll the probability of getting a 7 is $\frac{6}{36}$, since there are 6 ways to make a 7 with the two dice.

Notice that the number of possible outcomes for a whole craps game is infinite. Nevertheless, intuition suggests, ~~Cantell extrapolates~~ that "one sixth of all craps games are won with a 7 on the 1st roll".

or

$$\text{Prob } \{ 7 \text{ on 1st roll} \} = \frac{1}{6}.$$

$$\text{Similarly, } \text{Prob } \{ 11 \text{ on 1st roll} \} = \frac{2}{36}$$

Can you think of what the sample space and probability measure for the set of all craps games should be?

Let x_1, x_2, \dots, x_n $x_i = 1, 2, 3, 4, 5$ or 6

be a game of length n .

Then $\text{prob}(x_1, x_2, \dots, x_n) = p(x_1)p(x_2)\dots p(x_n)$

where $p(x_i)$ is the probability of rolling x_i in one roll.

e.g. $\text{Prob}\{\text{The game ends in one roll}\} = \frac{12}{36}$

$\text{Prob}\{\text{1st roll} = 4, \text{the game ends in 2 rolls}\} = \frac{27}{(36)^2}$

2	3	4	5	6	7
3		4	5	6	7
4		5	6	7	8
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

What is Prob { Win with point = 4 } ?

1st method (Summing a geometric series)

$$S = \sum_{k=1}^{\infty} \text{Prob} \{ \text{Win w/ point} = 4 \text{ on } k^{\text{th}} \text{ roll after 1st roll} \}$$

$$= \frac{3}{36} \sum_{k=1}^{\infty} \text{Prob} \{ \text{in } k \text{ rolls, 4 appears on } k^{\text{th}} \text{ roll with no previous 7's or 4's} \}$$

$$= \frac{3}{36} \left[\text{Prob} \{ \text{in a sequence of rolls, 4 appears before 7} \} \right]$$

Now, Prob { in one roll, anything but 4 or 7 }

$$= \frac{3}{4}$$

(see previous picture).

So,

$$S = \frac{3}{36} \sum_{k=1}^{\infty} \left(\frac{3}{4} \right)^{k-1} \cdot \frac{3}{36}$$

$$= \frac{3}{36} \left[\left(\sum_{k=0}^{\infty} \left(\frac{3}{4} \right)^k \right) \cdot \frac{3}{36} \right] = \frac{3}{36} \left[\frac{1}{1-\frac{3}{4}} \cdot \frac{3}{36} \right]$$

$$= \frac{3}{36} \left[\frac{1}{3} \right] = \frac{1}{36}.$$

2nd method (thinking)

The 7's and 4's make up 9 total outcomes, 3 of which are the 4's. So the first time a (7 or 4) hits, the probability that it's a 4 is $\frac{3}{9} = \frac{1}{3}$.

$$\therefore \text{Prob. } \{ \text{Win with point of 4} \} \\ = \left(\frac{3}{36} \right) \left(\frac{1}{3} \right) = \frac{1}{36}.$$

Prob $\{ \text{Win with a point of 5} \}$

$$= \left(\frac{4}{36} \right) \left(\frac{4}{10} \right) = \frac{1}{9} \cdot \frac{2}{5} = \frac{2}{45}$$

Prob $\{ \text{Win with a point of 6} \}$

$$= \left(\frac{5}{36} \right) \left(\frac{5}{11} \right)$$

Prob { Win with a point of 8 }

$$= \left(\frac{5}{36}\right)\left(\frac{5}{11}\right)$$

Prob { Win with a point of 9 }

$$= \frac{2}{45}$$

Prob { Win with a point of 10 }

$$= \frac{1}{36}$$

∴

Prob { Win at Craps }

$$= \frac{1}{6} + \frac{2}{36} + 2\left(\frac{1}{36} + \frac{2}{45} + \frac{5 \cdot 5}{36 \cdot 11}\right)$$

$$= .492929$$

What does this number mean?

Let's play a million games of Craps.

```
> g<-function(point){  
+  
+ x<-0  
+  
+ while(x!=7&x!=point){  
+  
+ x<-sample(1:6,1)+ sample(1:6,1)  
+  
+ if(x==7){0}else{1}  
+ }  
>  
>  
> craps<-function(y){  
+  
+ x<-sample(1:6,1) + sample(1:6,1)  
+  
+ if(x==7|x==11){1}else{ if (x==2|x==3|x==12){0}else{point<-x  
+ g(point)  
+  
+ }}}  
>  
>  
> d<-sapply(1:100000,craps)  
> sum(d)/length(d)  
[1] 0.49134  
> d<-sapply(1:100000,craps)  
> sum(d)/length(d)  
[1] 0.49192  
> d<-sapply(1:1000000,craps)  
> sum(d)/length(d)  
[1] 0.493385  
> sqrt(.492929*(1-.492929)*1000000)  
[1] 499.95  
> 493385-492929  
[1] 456  
>
```

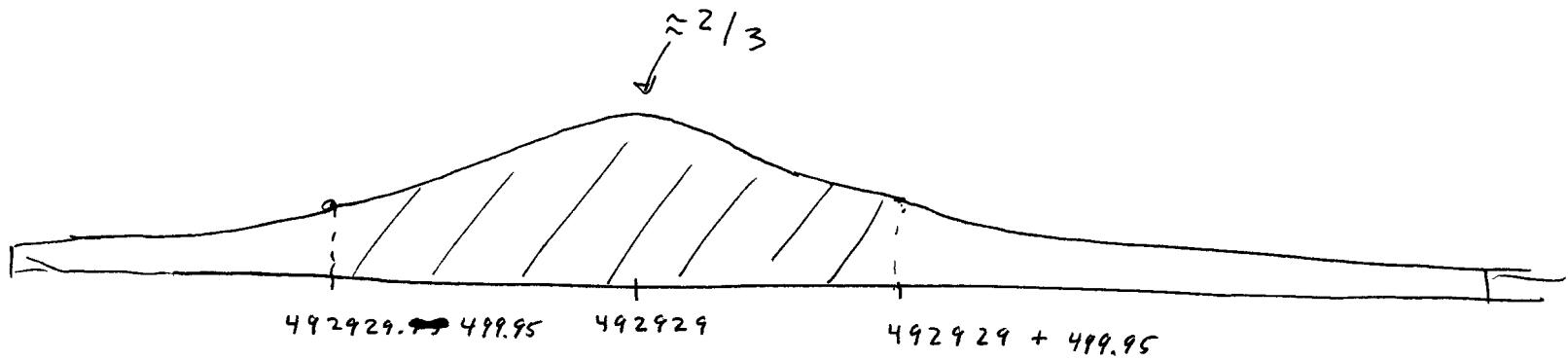
We were "expecting" 492929 wins out of 1 million tries, but in the simulation recorded here there were 493385.

Did I program it correctly?

Let S be the (random) number of wins in a million games of craps.

We shall see that (by the
Central limit theorem)

S is approximately normally distributed with mean value 492929 and standard deviation 499.95.



We expect about $\frac{1}{3}$ of all such observations to lie outside the interval $(492929 - 499.95, 492929 + 499.95)$

In the other direction, we expect about 95% of all such observations to lie in the interval

$$(492929 - (1.96)(499.95), 492929 + (1.96)(499.95))$$

(See the attached table of the normal distribution) (included in the text). Equivalently, we can expect a fluctuation this large or larger about one time in 20.

Another point of view is given

by the Law of Large Numbers.

As the number of Craps games in the simulation increases, the relative frequency of wins should approach .492929 with significant fluctuations from this becoming more and more rare.

For Next time

Read Chapter 1 of [BT]
and hand in Exercises 1, 2, 6, 14, 49
from [BT].