Math 137—Real Analysis.

MIDTERM PROBLEMS

WEDNESDAY, NOVEMBER 19, 2008

You may use books and your own notes. Please do not look on line for solutions and please work alone. Turn in what you have by Wednesday Nov. 26, and consider any problems you haven't solved by then as homework. After Nov. 26, you may collaborate.

1. (a) Prove the identity

$$\{y : y \in E_k \text{ for infinitely many k}\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k.$$

(b) Let A be the set of points $x \in (0,1)$ with the property that there are infinitely many fractions p/q with integers p, q, such that

$$|x - p/q| < 1/q^3$$

Show that A is a set of Lebesgue measure zero.

2. On the interval $[-1,1]$ consider the standard Banach Spaces L^1 and L^2 with the norms

$$\|f\|_{L^1} = \int_{-1}^{1} |f(x)|\, dx; \qquad \|f\|_{L^2} = \left(\int_{-1}^{1} |f(x)|^2\, dx \right)^{1/2}$$

Let $\{f_j\}_{j=1}^{\infty}$ denote a sequence of functions in L^2. Assume that $f_j \geq 0$,$\|f_j\|_{L_1} = 2$, and

$$|\|f_j\|_{L_2} - \sqrt{2}| \leq 2^{-j}.$$

Show that $\lim_{j\to\infty} f_j(x) = 1$ for almost every $x \in [-1,1]$.
Hints: Write $f_j = 1 + h_j$. Borel-Cantelli.

3. Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \leq f_n \leq 1$,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0$$

but the sequence $f_n(x)$ does not converge for any $x \in [-1,1]$.

4. Assume that (Ω, Σ, μ) is a measure space and $f \in L^p(\Omega)$ for some $0 < p < \infty$.

(a) Show that

$$\lim_{q\to\infty} \|f\|_{L^q} = \|f\|_{L^\infty}.$$

(b) Does the conclusion of part a) hold if we omit the assumption that $f \in L^p$? (proof or counterexample)

5. Let
$$F(x) = \int_{[0,\infty)} \frac{e^{-xt}}{|\sin t|^\alpha}\, dt, \qquad x > 0.$$
   (a) Find the values of the parameter α for which the function F is well-defined on $(0,\infty)$. (i.e. the integrand is in L^1.)
   (b) Show that F is infinitely diffentiable in $(0,\infty)$ for these values of α.

6. Let $\{f_n\}$ be a sequence of functions belonging to $L^1(\mathbb{R})$, the set of integrable functions on the real line $\mathbb{R}$. Suppose that there is a measureable function f so that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}$. Consider the following satements:
   (a) $\lim_{n\to} \int_{\mathbb{R}} |f_n(x)|\, dx = A$ exists
   (b) $\int_{\mathbb{R}} |f(x)|\, dx < +\infty$
   (c) $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|\, dx = \int_{\mathbb{R}} |f(x)|\, dx < +\infty$
   (d) $\lim_{n\to\infty} \int_{E} |f_n(x) - f(x)|\, dx = 0$ for every measurable set $E \subset \mathbb{R}$.

   Discuss (with proofs or examples) which of these four statement do or do not imply other statments in the list.

7. Let E be a proper subset of $\mathbb{R}$ which is non-empty and measurable. Assume that E is invariant under translation by rational numbers. Explicitly, this means that if $x \in E$ and r is any rational number, then $x + r \in E$. Show that either E has Lebesgue measure zero or that $\mathbb{R} - E$ has Lebesgue measure zero. Give examples to show that both conclusions are possible.

8. (a) Let $E, F \subset \mathbb{R}^N$ be Lebesgue measurable subsets both of which have finite and positive measure. For $x \in \mathbb{R}^N$ we define the translates $E_x = \{x + y : y \in E\}$. Show that the set
$$G = \{x \in \mathbb{R}^N : E_x \cap F \neq \emptyset\}$$
has positive Lebesgue measure. (Hint: What does the Lebesgue differention theorem say when applied to the characteristic function of a set?)
   (b) If $E \subset \mathbb{R}$ is a dense and open subset of the real line, must its Lebesgue measure be infinite? (proof or counterexample).

9. For $f \in L^1_{loc}(\mathbb{R}), j \geq 0$ define the so-called "conditional expectation operator" E_j by
$$E_j f(x) = 2^j \int_{n2^{-j}}^{(n+1)2^{-j}} f(\xi)\, d\xi \qquad \text{if } x \in [n2^{-j}, (n+1)2^{-j}), \qquad n \in \mathbb{Z}$$
   (a) Show that for $f \in L^1(\mathbb{R})$ we have
$$\lim_{j\to\infty} E_j f(x) = f(x) \text{ almost everywhere.}$$
   (b) Show that for $f \in L^2(\mathbb{R})$ one has
$$\lim_{j\to\infty} \|E_j f - f\|_{L^2} = 0.$$

10. Let $1 \leq p < \infty$, and let $f_n \in L^p(\mathbb{R})$ be a sequence of functions. Suppose
$$\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_{L^p} < \infty.$$
Show that the sequence f_n converges pointwise almost everywhere.

11. Read Levy's treatment of the Strong Law of Large Numbers in section 1.4 of [PT]. Then do problems 1.4.27, 1.4.28 and 1.4.30 in [PT].