Math 137 Dec 3 2008.

$\{X_n\}_{n=1}^{\infty}$ i.i.d. $E(X_n)=0$

$E(X_n^2) = Var(X_n) = 1$.

$$S_n = \sum_{k=1}^{n} X_k$$

Thm: (LIL). $\overline{\lim_{n\to\infty}} \dfrac{S_n}{\sqrt{2n \log\log n}} = 1$ a.s.

$\Rightarrow \hat{S}_n \equiv \dfrac{S_n}{\sqrt{n}}$ diverges. a.s.

but $E(\hat{S}_n) = 0$
$Var(\hat{S}_n) = 1$ $\forall n$.

and this implies that

$$E\left(a(\hat{S}_n)^2 + b(\hat{S}_n) + c \right)$$

is constant $= a + c$.

ind. of n or of the $\{X_n\}$.

i.e. $E(\varphi(\hat{S}_n))$ is const

for $\varphi(x) = ax^2+bx+c$.

and we will show that the statement also holds for a much wider class of fcns φ.

Some heuristic motivation:

If X_1 has moments of all orders what is $\lim\limits_{n\to\infty} E^P[(\hat{S}_n)^m]$

for a fixed $m \geq 3$?

$$E^P[S_n^{m+1}] = \sum_{i=1}^n E(X_i(S_n^m))$$

$$= n\,E(X_n(S_{n-1}+X_n)^m)$$

$$= n\,E\left(X_n \sum_{j=0}^m \binom{m}{j} X_n^j S_{n-1}^{m-j}\right)$$

$$= \cancel{n\,E}\;\; n \sum_{j=0}^{m} \binom{m}{j} E\left(X_n^{j+1} S_{n-1}^{m-j}\right)$$

$$= n \sum_{j=0}^{m} \binom{m}{j} E\left(X_n^{j+1}\right) E\left(S_{n-1}^{m-j}\right)$$

$$= n \cdot m\, E\left(S_{n-1}^{m-1}\right) + n \sum_{j=2}^{m} \binom{m}{j} E\left(X_n^{j+1}\right) E\left(S_{n-1}^{m-j}\right)$$

$$= n \cdot m\, E\left(S_{n-1}^{m-1}\right) + n \sum_{j=2}^{m} \binom{m}{j} E\left(X_n^{j+1}\right) E\left(S_{n-1}^{m-j}\right)$$

$$\therefore \quad E^P\Bigg[\Big(\underbrace{\frac{S_n}{\sqrt{n}}}_{\hat{S}_n}\Big)^{m+1}\Bigg] = \frac{n \cdot m}{n} E\left(\left(\frac{S_{n-1}}{\sqrt{n}}\right)^{m-1}\right)$$

$$\cancel{+ \left(\frac{n}{(\sqrt{n})^{m+1}}\right) \sum_{j=2}^{m}}$$

$$+ \; n \sum_{j=2}^{m} \binom{m}{j} E\left(X_n^{j+1}\right) \frac{1}{(\sqrt{n})^{j+1}} E\left(\left(\frac{S_{n-1}}{\sqrt{n}}\right)^{m-j}\right)$$

<u>Claim:</u> With $L_m \equiv \lim_{n \to \infty} E^P\left[\hat{S}_n^{m}\right]$

we have $L_{m+1} = m L_{m-1}$.

Pf: $L_0 = 1$.

$L_1 = 0$ so the claim holds for

$L_2 = 1$. $m = 1$.

Given the claim is true for m

we have from the above.

$$L_{m+1} = m L_{m-1} + \lim_{n\to\infty} \left(\frac{1}{n^{3/2}}\right) \cdot \sum_{j=2}^{m} \binom{m}{j} E(X_1^{j+1}) \cdot \frac{1}{(\sqrt{n})^{j-1}} \underbrace{E\left(\left(\frac{S_{n-1}}{\sqrt{n}}\right)^{m-j}\right)}_{\Downarrow \; L_{m-j}}$$

$\therefore$

$$L_{m+1} = m L_{m-1}.$$

$\blacksquare$

and we must have.

$$L_{2m-1} = 0. \qquad L_{2m} = \prod_{\ell=1}^{m} (2\ell-1)!$$

$$L_{2m} = \frac{(2m)!}{(2m)(2(m-1))(\dots)2} = \frac{(2m)!}{2^m m!}$$

We then see that.

$\lim_{n\to\infty} E^P(\varphi(\hat{S}_n))$ does not depend on the choice of the X_i.

whenever φ is a polynomial.

We are conjecturing that this will remain true for a wide class of functions φ. (i.e. independence of the limit from the particular X_i)

If our conjecture is valid we should be able to guess the limit with a convenient choice of the X_i.

But if $X_1 \sim N(0,1)$ then

$$\hat{S}_n \sim N(0,1).$$

So $E(\varphi(\hat{S}_n)) = \int_{\mathbb{R}} \varphi(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

$\forall n$.

Note that if X_1 is a r.v. with the property that

$$E[X_1^m] = \lim_{n\to\infty} E[\hat{S}_n^m] = \frac{(2m)!}{2^{m/2}(\frac{m}{2})!} \quad m \text{ even}$$
$$= 0 \text{ else.}$$

then.

$$E(e^{\alpha X_1}) = E\left(\sum_{\ell=0}^{\infty} \frac{(\alpha X_1)^\ell}{\ell!}\right)$$

(assuming e.g. that $e^{\alpha|X_1|}$ is integrable)

$$= \sum_{\ell=0}^{\infty} \frac{\alpha^\ell}{\ell!} E(X_1^\ell).$$

$$= \sum_{\ell=0}^{\infty} \frac{\alpha^{2\ell}}{2^\ell \ell!} = e^{\alpha^2/2}$$

i.e. X_1 must be $N(0,1)$.

So our conjecture is that

$$\lim_{n\to\infty} E^P[\varphi(\hat{S}_n)] = \int_{\mathbb{R}} \varphi(y) \tfrac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

for a large class of i.i.d $\{X_k\}$ and a large class of functions φ.

$(\Omega, \mathcal{F}, P)$ a prob. space.

$\{X_m\}_1^n$ ind.

$E(X_m) = 0$.

$+\infty > E^P(X_m^2)^{1/2} = \sigma_m > 0$

Put $S_n \equiv \sum_1^n X_m$

$\Sigma_n \equiv \left(\sum_1^n \sigma_m^2\right)^{1/2}$

$\hat{S}_n = \frac{S_n}{\Sigma_n}$.

Thm: $\varphi \in C^3(\mathbb{R}, \mathbb{R})$.

$|\varphi''| < C_2$

$|\varphi'''| < C_3$.

given $\epsilon > 0$,

$$\left| E^P[\varphi(\hat{S}_n)] - \int_{\mathbb{R}} \varphi(x) \tfrac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right|$$

$$\leq \left(\frac{\epsilon}{6} + \frac{r_n}{2}\right) \|\varphi'''\|_u + g_n(\epsilon) \|\varphi''\|_u.$$

In particular,

Since $r_n^2 \leq \epsilon^2 + g_n(\epsilon)$, $\epsilon > 0$.

we have.

$$\lim_{n\to\infty} E^P[\varphi(\hat{S}_n)] = \int_{\mathbb{R}} \varphi(y) \tfrac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Lindberg condition [if $g_n(\epsilon) \to 0$ as $n \to \infty$ for each $\epsilon > 0$.

if the X_k are i.i.d ~~mean zero~~, var $= 1$. this is as above.

let
$$r_n = \max_{1 \le m \le n} \frac{\sigma_m}{\Sigma_n}.$$

$$g_n(\epsilon) = \frac{1}{\Sigma_n^2} \sum_{m=1}^{n} E^P\left[X_m^2 ; |X_m| \ge \epsilon \Sigma_n \right]$$

$\epsilon > 0$.

in the i.i.d case ~~$\mu = 0, \sigma^2 = 1$~~. this is.

$$r_n = \frac{1}{\sqrt{n}}$$

$$g_n(\epsilon) = \frac{1}{\sigma_1^2} E^P\left[X_1^2 , |X_1| \ge \sqrt{n}\, \sigma_1 \epsilon \right].$$

$\longrightarrow 0$ as $n \to \infty$

for each ϵ.