Math 137 12/1/08.

Thm: f real valued, msble on $E \subset \mathbb{R}^N$ $\underbrace{\quad}_{\text{works on a } \sigma\text{-finite measure space.}}$

let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$

Then

$$\|f\|_p = \sup \left\{ \int_E fg : \begin{array}{l} g \text{ real valued} \\ \|g\|_{p'} \le 1 \\ \int_E fg \text{ exists.} \end{array} \right\}$$

Pf:

$$\int_E fg \le \left(\int_E f^p\right)^{1/p} \left(\int_E g^q\right)^{1/q}$$

$$\le \|f\|_p \quad \text{since } \|g\|_q \le 1.$$

$$\therefore \sup \int_E fg \le \|f\|_p.$$

For the opposite inequality, first suppose that

$f \ge 0$, $1 < p < \infty$.

If $\|f\|_p = 0$ then $f = 0$ a.e. ✓.

If $0 < \|f\|_p < \infty$ then w.m.a.

$\|f\|_p = 1$. after dividing by $\|f\|_p$.

then

$$\int_E f\, f^{p-1}\, d\mu = 1.$$

with $g = f^{p-1}$

$$g^{p'} = f^{(p-1)p'} = f^p$$

$\frac{1}{p} + \frac{1}{p'} = 1.$

$1 + \frac{p}{p'} = p.$

so $\|g\|_{p'} = 1.$

$p - 1 = \frac{p}{p'}$

If $\|f\|_p = +\infty$

let $f_k = \begin{cases} 0 & |\vec{x}| > k \\ \min[f(x), k] & |\vec{x}| \leq k \end{cases}$

then $f_k \in L^p$

and $\|f_k\|_p \to \|f\|_p = +\infty.$

$\exists\ g_k \in L^{p'}$ s.t. $g_k \geq 0$ and $\|g_k\|_{p'} = 1$

$$\int f_k g_k = \|f_k\|_p$$

Since $f \geq f_k$ we have

$$\int_E f g_k \geq \int_E f_k g_k = \|f_k\|_p \to +\infty$$

so that. $\sup_{\|g\|_{p'}=1} \int_E fg = +\infty = \|f\|_p$.

To get by the restriction that $f \geq 0$.

take $g_k \geq 0$ s.t.

$$\int_E |f| g_k \to \|f\|_p.$$

and note that

$$\int_E |f| g_k = \int_E f \cdot \text{sign}(f)\, g_k.$$

where $\text{sign}(f) = \begin{cases} 1 & f \geq 0 \\ -1 & f \leq 0. \end{cases}$

Since $|(\text{sign} f) g_k| = |g_k| = g_k$.

the result follows.

If $f \in L^1$ then $\mathrm{sign}(f) \in L^\infty$.

If $f \in L^\infty$, ~~bdd~~

suppose $f \geq 0$.

let $A_n = \{x : f(x) \geq \|f\|_\infty - 1/n\} \cap \{|\vec{x}| \leq M_n\}$

where M_n is chosen so that

A_n has positive measure.

then $g_n = \dfrac{\chi_{A_n}}{|A_n|} \in L^1$. $\|g_n\|_{L_1} = 1$.

$\|f\|_\infty \geq$ and $\displaystyle\int_E f \, \frac{\chi_{A_n}}{|A_n|} \geq (\|f\|_\infty - \tfrac{1}{n})$.

Large Deviations (Preliminaries).
(for sums of independent random variables).

Suppose $X_1, \ldots, X_n$ are i.i.d. $(\Omega, P, \mathcal{F})$.

Let μ be their common distribution

i.e. if $E \subset \mathbb{R}$ is Borel then

$$\mu(E) = P(\omega \in \Omega : X_1(\omega) \in E).$$

Suppose also that

$$M_\mu(\xi) \equiv \int_{\mathbb{R}} e^{\xi x} \mu(dx) < +\infty \quad \forall \xi \in \mathbb{R}.$$

e.g. $\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ (standard normal)

has this property.

Note that $M_\mu(\xi)$ is infinitely differentiable.

PF: Fix ξ. $\displaystyle \lim_{h \to 0} \frac{M_\mu(\xi+h) - M_\mu(\xi)}{h}$

$$= \lim_{h \to 0} \int_{\mathbb{R}} \frac{e^{(\xi+h)x} - e^{\xi x}}{h} \, d\mu.$$

$$= \lim_{h\to 0} \int_{\mathbb{R}} \cancel{x}\, e^{\xi x} \left(\frac{e^{hx}-1}{hx} \right) d\mu.$$

out
$$\left\{ \begin{aligned} \left| \frac{e^{hx}-1}{hx} \right| &\le C \quad \text{if } |hx| \le 1 \\ &\le |e^{hx}-1| \quad \text{if } |hx| \ge 1. \end{aligned} \right.$$

$h \to 0^+$ and assume $h \le \epsilon$.

$\Rightarrow$ if $x < 0$. then $\left| \frac{e^{hx}-1}{hx} \right| < 1$.

if $x > 0$ then

$$0 < \frac{e^{hx}-1}{hx} \le C_M \quad \text{for } hx \le M.$$

and if $hx \ge M > 1$.

$$\frac{e^{hx}-1}{hx} \le e^{hx} \le e^{\epsilon x}.$$

(and similarly for $\lim_{h\to 0^-}$.)

Since $\int x\, e^{\xi x} \begin{cases} e^{\epsilon x} \\ C \end{cases} d\mu < +\infty$.

L.D.C.T $\Rightarrow$ $M_\mu'(\xi)$ exists. $\forall \xi \in \mathbb{R}$.

and

$$M_\mu'(\xi) = \int_{\mathbb{R}} x e^{\xi x} \mu(dx) \quad \forall \xi \in \mathbb{R}$$

The above argument can be repeated so

$\forall n \in \mathbb{Z}^+$

$$M_\mu^{(n)}(\xi) = \int_{\mathbb{R}} x^n e^{\xi x} \mu(dx).$$

and $\therefore$

$$M_\mu^{(n)}(0) = \int_{\mathbb{R}} x^n \mu(dx) = \int_\Omega (X^n)\, dP$$

$$= E_P(X^n).$$

if X is standard normal.

then

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

and $$M_\mu(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\xi x} e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\left(\frac{x^2 - 2\xi x + \xi^2}{2}\right)} e^{\xi^2/2}\, dx$$

$$= e^{\xi^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{2}}\, dx$$

$$= e^{\xi^2/2}.$$

A r.v. Y is said to have an $N(\mu, \sigma^2)$ distribution if

$\frac{Y-\mu}{\sigma}$ has the standard normal distribution

i.e. $\forall \Gamma \in \mathcal{B}_{\mathbb{R}}$.

$$P(Y \in \Gamma) = \int_\Gamma \frac{1}{\sigma} \gamma\left(\frac{y-m}{\sigma}\right) dy.$$

$$\gamma(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}.$$

if $Y \sim N(\mu, \sigma^2)$

then $Y = \sigma X + \mu$ where $X \sim N(0,1)$.

and

$$E(e^{\xi Y}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\xi(\sigma X + \mu)} \cdot e^{-x^2/2} \, dx.$$

$$= e^{\mu \xi} \, e^{(\xi\sigma)^2/2}.$$

In exercises 1.2.17, 1.2.19 we saw that a prob. law $d\mu$ for which

$$M_\mu(\xi) = \int e^{\xi x} d\mu(x) < +\infty \quad \forall \xi$$

can be recovered from $M_\mu(\xi)$.

If X_1 is $\sim N(m_1, \sigma_1^2)$ and $X_2 \sim N(m_2, \sigma_2^2)$ ~~then~~. and X_1, X_2 are independent then.

$$E\left(e^{\xi(X_1+X_2)}\right) = E(e^{\xi X_1}) E(e^{\xi X_2}) \quad \text{by independence}$$

$$= e^{m_1 \xi} e^{(\sigma_1 \xi)^2/2} \; e^{m_2 \xi} e^{(\sigma_2 \xi)^2/2}$$

$$= e^{(m_1+m_2)\xi} \; e^{\frac{(\sigma_1^2+\sigma_2^2)\xi^2}{2}}$$

So $X_1 + X_2$ is $\sim N(m_1+m_2, \sigma_1^2+\sigma_2^2)$.

and e.g.

if $X_1, \ldots, X_n$ are ind. $N(0,1)$

then $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(0, \frac{1}{n})$.

$$\therefore \quad P(\bar{S}_n \in \Gamma) = \sqrt{\frac{n}{2\pi}} \int_\Gamma e^{-\left(\frac{n|y|^2}{2}\right)} dy.$$

We know from our exercises that if

$f \in L^p(\cancel{\mu}, E)$ for some $p > 1$ then

$$\lim_{q\to\infty} \int_E |f|^q \, d\cancel{\mu} \to \|f\|_\infty.$$

$$\therefore \left(\int_\Gamma (e^{-y^2/2})^n \, dy \right)^{1/n} \to \operatorname{ess\,sup}\{e^{-y^2/2} : y \in \Gamma\}$$

$$\therefore \quad P(\bar{S}_n \in \Gamma)^{1/n} \to \operatorname{ess\,sup}\{e^{-y^2/2} : y \in \Gamma\}.$$

so $\frac{1}{n}\log\left[P(\bar{S}_n \in \Gamma)\right] \to -\operatorname{ess\,inf}\{\frac{|y|^2}{2} : y \in \Gamma\}$

$$\overset{\gamma(x)}{\left(1 - \frac{3}{x^4}\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-x^2/2}} < \gamma(x) < \left(1 + \frac{1}{x^2}\right)\gamma(x).$$

and.
$$\gamma'(x) = -x\,\gamma(x).$$

$$\frac{d}{dx}\left(\left(\frac{1}{x} - \frac{1}{x^3}\right)\gamma(x)\right) = \left(-\frac{1}{x^2} + \frac{3}{x^4}\right)\gamma(x)$$

$$+ \left(\frac{1}{x} - \frac{1}{x^3}\right)\left(-x\gamma(x)\right).$$

$$= \gamma(x)\left(-\frac{1}{x^2} + \frac{3}{x^4} - 1 + \frac{1}{x^2}\right)$$

$$= -\left(1 - \frac{3}{x^4}\right)\gamma(x).$$

$$\frac{d}{dx}\left(1/x\ \gamma(x)\right) = -\frac{1}{x^2}\gamma(x) + \frac{1}{x}\left(-x\gamma(x)\right)$$

$$= -\left(1 + \frac{1}{x^2}\right)\gamma(x).$$

$$\int_x^\infty \left(1 - \frac{3}{y^4}\right)\cdot\gamma(y)\,dy < \int_x^\infty \gamma(y)\,dy < \int_x^\infty \left(1 + \frac{1}{y^2}\right)\gamma(y)\,dy.$$

$$\Rightarrow \left(\frac{1}{x} - \frac{1}{x^3}\right)\cancel{\left(1 - \frac{3}{x^4}\right)}\gamma(x) < \int_x^\infty \gamma(y)\,dy < \frac{1}{x}\gamma(x).$$

So

$$P(|\bar{S}_n| \geq \epsilon.) = P(\sqrt{n}\,|\bar{S}_n| \geq \epsilon\sqrt{n}).$$

$$\leq 2\cdot \frac{1}{\epsilon\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\epsilon\sqrt{n})^2}{2}}$$

$$= \sqrt{\frac{2}{n\pi\epsilon^2}}\, e^{-\frac{n\epsilon^2}{2}} \qquad \epsilon > 0.$$

and if $0 < \epsilon < |a|$ and n is sufficiently large.

$$P(|\bar{S}_n - a| < \epsilon) = P(|\sqrt{n}\,\bar{S}_n - a\sqrt{n}| < \epsilon\sqrt{n}).$$

$$P\left(-\epsilon\sqrt{n} < \sqrt{n}\,\bar{S}_n - a\sqrt{n} < \epsilon\sqrt{n}.\right)$$

$$= P\left(\sqrt{n}\,\bar{S}_n > a\sqrt{n} - \epsilon\sqrt{n}\right) - P\left(\sqrt{n}\,\bar{S}_n > a\sqrt{n} + \epsilon\sqrt{n}\right)$$

assume $a > 0.$

$0 < \epsilon < a.$

$\vdots$

$$\geq \left(\frac{1}{(a-\epsilon)\sqrt{n}} - \frac{1}{(a-\epsilon)^3 n^{3/2}}\right)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{n(a-\epsilon)^2}{2}}$$

$$- \frac{1}{(a+\epsilon)\sqrt{n}}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{n}{2}\cancel{(a+\epsilon)^2}\,[(a-\epsilon)^2+4a\epsilon]}.$$

$$= \frac{(a+\epsilon)\sqrt{n} - e^{-2na\epsilon}(a-\epsilon)\sqrt{n}}{(a^2-\epsilon^2)\,n}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{n}{2}(a-\epsilon)^2}$$

$$- \frac{1}{(a-\epsilon)^3 n^{3/2}}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{n}{2}(a-\epsilon)^2}.$$

$$\geq \frac{a}{a^2-\epsilon^2\ \sqrt{n}}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{n}{2}(a-\epsilon)^2}.$$ large n.

In general if X_n's are $N(m, \sigma^2)$ (independent) then

$$P\left(|\bar{S}_n - m| \geq \sigma\epsilon\right) \leq \sqrt{\frac{2}{n\pi\epsilon^2}}\, e^{-\frac{n\epsilon^2}{2}} \qquad \epsilon > 0.$$

and $\quad 0 < \epsilon < |a|$, n large

$$\Rightarrow$$

$$P\left(|\bar{S}_n - (m+a)| < \sigma\epsilon\right) \geq \frac{a}{(a^2-\epsilon^2)\sqrt{2\pi}\sqrt{n}}\, e^{-n\frac{(|a|+\epsilon)^2}{2}}.$$

$$\geq \frac{a}{a^2-\epsilon^2}\,\frac{1}{\sqrt{2\pi}\sqrt{n}}\, e^{-n\frac{(|a|-\epsilon)^2}{2}}.$$

The idea of Cramer's Theorem is that for sequences $\{X_n\}$ of P-iid. r.v's. with common distribution μ s.t.

$$M_\mu(\xi) = \int_{\mathbb{R}} e^{\xi x}\, d\mu(x) < +\infty \quad \forall\, \xi \in \mathbb{R}.$$

we can bounds which are similar in appearance to the above.

If the X_i are independent and i.i.d. (μ)

then $$E\left(e^{\xi(X_1+\dots+X_n)}\right) = (M_\mu(\xi))^n$$

and (for $\xi \ge 0$)

$$P(\bar{S}_n \ge a) = P(S_n \ge na) = P(e^{\xi S_n} \ge e^{\xi n a})$$

$$\le e^{-\xi n a} E[e^{\xi S_n}; \bar{S}_n \ge a]$$

$$\le e^{-\xi n a} M_\mu(\xi)^n.$$

$$= e^{-(\xi n a - n \log M_\mu(\xi))}.$$

$$= e^{-n(\xi a - \underbrace{\log M_\mu(\xi)}_{\Lambda_\mu(\xi)})}.$$

So

$$P(\bar{S}_n \ge a) \le e^{-n \sup_{\xi \in [0,\infty)} (\xi a - \Lambda_\mu(\xi))}.$$

<u>remark:</u> if $(N(m,\sigma^2))$ $M_\mu(\xi) = e^{\xi m + \frac{\sigma^2 \xi^2}{2}}$

then $\log M_\mu(\xi) = \xi m + \frac{\sigma^2\xi^2}{2}$

and we get $P(\bar{S}_n \ge m+\epsilon) \le e^{-n(\sup_{\xi\in[0,\infty)}(\xi\epsilon - \frac{\sigma^2\xi^2}{2}))}$

$$\sup_{\xi \in (0,\infty)} \left(\xi \left(\epsilon - \frac{\sigma^2}{2} \xi \right) \right)$$

occurs at $\dfrac{\epsilon}{\sigma^2}$.

and equals $\dfrac{\epsilon^2}{2\sigma^2}$

so $P(\bar{S}_n \geq m+\epsilon) \leq e^{-\frac{n\epsilon^2}{2\sigma^2}}$.

which is (for practical purposes) the same as the bound before.

Def:

$$I_\mu(x) \equiv \sup \{ \xi x - \Lambda_\mu(\xi) : \xi \in \mathbb{R} \}$$

is the Legendre Transform of Λ_μ.