Math 137 11/24/08.

1.4.24 X_n ind. $E(X_n)=0$ $\sum_{n=1}^{\infty} E(X_n^2) < +\infty$.

$S_n \to S$ a.s.

$$\int \underline{\lim} |S_n|^2 \, dP \le \underline{\lim} \int |S_n|^2 \, dP.$$

$$\Rightarrow \int |S|^2 \, dP < +\infty. \quad \text{so } S \in L^2.$$

$$\int |S_n - S|^2 \, dP = \int \underline{\lim}_{N\to\infty} \Big| \sum_{m=n+1}^{N} X_m \Big|^2 dP.$$

$$\le \underline{\lim}_{N\to\infty} \int \Big| \sum_{m=n+1}^{N} S_m \Big|^2 dP.$$

$$\to 0 \quad n \to \infty.$$

(orthogonality)

$$P\left(\sup_{n\ge 1} |S_n|^2 \ge t \right) \le \frac{1}{t} \lim_{N\to\infty} [$$

$$E^P\Big[S_N^2, \max_{1\le n\le N} |S_n|^2 > t\Big]$$

$$E^P\Big[S_N^2, \max_{1\le n\le N} |S_n|^2 > t\Big] + E\Big[S_N^2, \max_{1\le n\le N} |S_n|^2 \le t\Big] = E_P[S_N^2] \to E_P[S^2].$$

$\to E[S^2, \sup_{n\ge 1} |S_n^2| \le t]$.

Inequalities

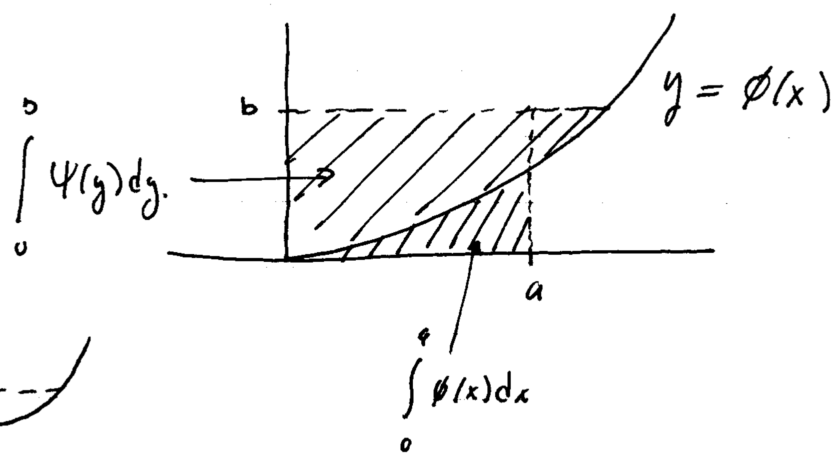
Thm: let $y = \phi(x)$ be cont., real vld. strictly increasing for $x \geq 0$ and let $\phi(0) = 0$.

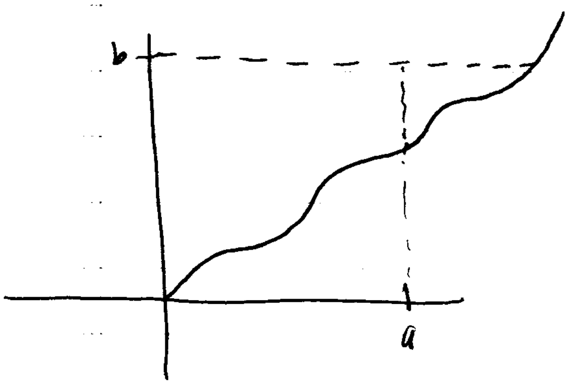
If $x = \psi(y)$ is the inverse of ϕ then for $a, b > 0$.

$$ab \leq \int_0^a \phi(x)\,dx + \int_0^b \psi(y)\,dy.$$

with equality iff $b = \phi(a)$.

Pf:





If $\phi(x) = x^\alpha \quad \alpha > 0$

then $\psi(y) = y^{1/\alpha}$ and we have.

$$ab \leq \int_0^a x^\alpha dx + \int_0^b y^{1/\alpha} dy$$

$$= \frac{a^{\alpha+1}}{\alpha+1} + \frac{b^{1/\alpha + 1}}{1 + 1/\alpha} \qquad a, b \geq 0$$

put $p = \alpha + 1$, $p' = \frac{1}{\alpha} + 1$ to get

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

if $a, b \geq 0$
$1 < p < +\infty$ and.
$\frac{1}{p} + \frac{1}{p'} = 1$.

$p, p' > 1$ s.t $\frac{1}{p} + \frac{1}{p'} = 1$ are call

conjugate exponents.

take $p' = \infty$ if $p = 1$ $\quad p' = 1$ if $p = \infty$

and $\therefore \int |f_1 g_1| \, d\mu \le 1$

so $\int |f g| \, d\mu \le \|f\|_p \|g\|_q$.

$p = q = 2$ gives the Cauchy Schwarz inequality.

$$\int_E |fg| \le \left(\int |f|^2\right)^{1/2} \left(\int |g|^2\right)^{1/2}.$$

<u>Thm:</u> <u>Minkowski's inequality</u>

if $1 \le p \le \infty$ then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

<u>Pf:</u> if $p = 1$.

$$\int_E |f+g| \, d\mu \le \int_E |f| + |g| \, d\mu \quad \checkmark$$

if $p = \infty$ then $|f| \le \|f\|_\infty$ a.e.

$|g| \le \|g\|_\infty$ a.e.

so $|f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ a.e.

$\therefore \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

Thm: <u>Holder's</u> <u>inequality</u>

If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad \text{i.e.}$$

$$\int_E |fg|\, d\mu \leq \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^q\right)^{1/q}.$$

and

$$\int_E |fg| \leq \|f\|_\infty \int_E |g|.$$

<u>Pf:</u> ($p=\infty$ case is clear).

Suppose $1 < p < \infty$ and $\|f\|_p = \|g\|_q = 1$.

then

$$\int |fg|\, d\mu \leq \int \left(\frac{|f|^p}{p} + \frac{|g|^q}{q}\right) d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

If $\|f\|_p = 0$ (or $\|g\|_q = 0$) then ($f=0$ a.e) or ($g=0$ a.e)

and the inequality is trivial.

so, in general, take

$$f_1 = \frac{f}{\|f\|_p} \qquad g_1 = \frac{g}{\|g\|_q} \qquad \text{then } \|f_1\|_p = 1,\ \|g_1\|_q = 1.$$

Pf: cont. for $1 < p < \infty$

$$\|f+g\|_p^p = \int |f+g|^p \, d\mu = \int |f+g|^{p-1} |f+g| \, d\mu$$

$$\leq \int |f+g|^{p-1} |f| \, d\mu + \int |f+g|^{p-1} |g| \, d\mu.$$

for $p>1$ $\left(\frac{p}{p-1}, p.\right)$ are conjugate exponents.

$$\leq \left(\int |f+g|^p d\mu\right)^{\frac{p-1}{p}} \left(\int |f|^p d\mu\right)^{1/p} + \left(\int |f+g|^p d\mu\right)^{\frac{p-1}{p}} \left(\int |g|^p d\mu\right)^{1/p}$$

$$\therefore \quad \|f+g\|_p^p \leq \|f+g\|_p^{p-1} \left(\|f\|_p + \|g\|_p \right).$$

$$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

(w.m.a. $\|f+g\|_p \neq 0$).

Note that if $f, g \in L^p(E)$ then $f+g \in L^p$ since.

$$|f+g|^p \leq |2 \max(f,g)|^p = 2^p |\max(f,g)|^p$$

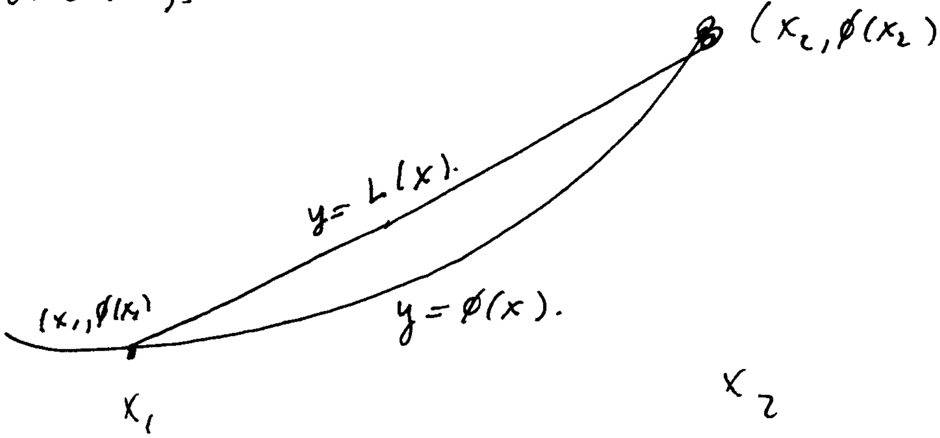
$$\leq 2^p (|f| + |g|)^p.$$

So if $\|f+g\|_p = +\infty$ then either $\|f\|_p = +\infty$ or $\|g\|_p = +\infty$.

Jensen's inequality

↳ is often stated for convex fcns.

def: ϕ, defined and finite on an interval (a,b) is said to be convex in (a,b) if for every $[x_1, x_2] \subset (a,b)$, the graph of ϕ on $[x_1, x_2]$ lies on or below the line segment connecting $(x_1, \phi(x_1))$ $(x_2, \phi(x_2))$.



$$\phi(tx_1 + (1-t)x_2) \le L(tx_1 + (1-t)x_2) \quad 0 \le t \le 1$$

$$= t\phi(x_1) + (1-t)\phi(x_2).$$

then

$$E(g(X)) \geq g(E(X)).$$

Pf: Since g is convex, the graph of g lies above each of its supporting lines. So for each x_0 there is a const. c s.t.

$$g(X) \geq g(x_0) + c(X - x_0).$$

Let $x_0 = E(X)$ then

$$g(X) \geq g(E(x_0)) + c(X - E(X)).$$

$$\therefore \quad E(g(x)) \geq g(E(x_0)) \quad \blacksquare$$

Applying the last thm to simple fcns ≥ 0 on Ω or just using the defn inductively

$$\phi\left(\sum_{j=1}^{m} t_j x_j \right) \leq \sum_{j=1}^{m} t_j \, \phi(x_j)$$

$\{x_j\} \subset (a,b)$ when $\sum t_j = 1$, $t_j \geq 0$ and ϕ is convex on (a,b)

e.g. If $p_j > 1$ and $\sum_{j=1}^{N} \frac{1}{p_j} = 1$

then

$$\exp\left(\sum_{j=1}^{m} \frac{x_j}{p_j} \right) \leq \sum_{j=1}^{m} \frac{1}{p_j} e^{x_j}$$

Let $a_j = e^{x_j/p_j}$

then $\log a_j = \frac{x_j}{p_j}$ so

$$\exp\left(\sum_{j=1}^{N} \log a_j \right) \leq \sum_{j=1}^{m} \frac{1}{p_j} a_j^{p_j}$$

and we have $a_1 \cdots a_N \leq \sum_{j=1}^{m} \frac{a_j^{p_j}}{p_j}$.

It follows that

$$\int_E (f_1, \dots f_N)\, d\mu \le \|f_1\|_{p_1} \cdots \|f_N\|_{p_N}$$

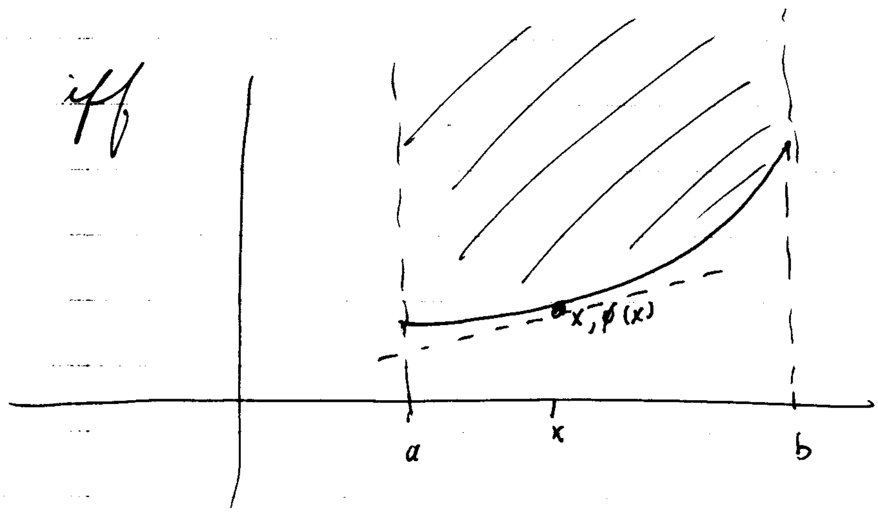
when $\displaystyle\sum_{j=1}^{N} \frac{1}{p_j} = 1 \quad p_j > 1.$

So ϕ is convex in (a,b) iff

$$\phi(tx_1+(1-t)x_2) \le t\phi(x_1)+(1-t)\phi(x_2)$$

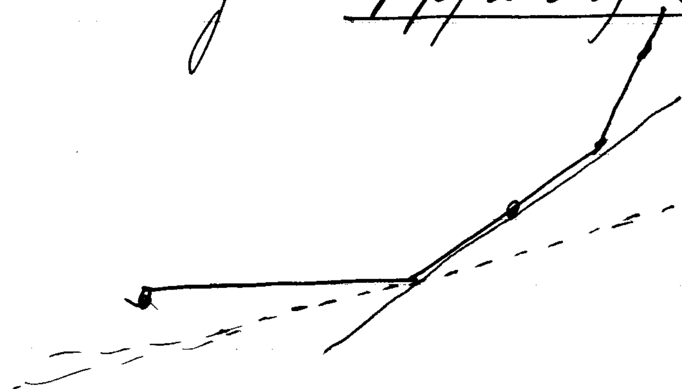
for all $a<x_1<x_2<b$, $0\le t\le 1$

iff



$\{a\le x\le b,\ y\ge\phi(x))$ is a convex set.

iff every *supporting line* at $(x,\phi(x))$ with lies on or below the graph of ϕ.



Jensen's *Inequality* (1st Form)

if X is a r.v. on a prob. space $(\Omega,P,\mathcal{F})$

and g is a convex fcn.