Math 37 11/19/08

last time.

$f \in L^1_{loc}(\mathbb{R}^n)$

$$\Rightarrow \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy = f(x)$$

for a.e. x.

<u>Lebesgue set of f</u>

$\mathcal{F}$ a family of msble subsets of $\mathbb{R}^n$.

$\mathcal{F}$ is <u>regular</u> if

$\exists\ c > 0$ s.t.

$S \in \mathcal{F} \Rightarrow \exists\ B = B(0,r)$ for some r

s.t.

$S \subset B$ and $m(S) \geq c\, m(B)$.

<u>e.g.</u> $\mathcal{F} = \{ \delta U : \delta > 0,\ U$ fixed, bdd. $0 < m(U) \}$.

$\mathcal{F} = \{ \text{cube } Q : d(Q,0) \leq C \operatorname{diam}(Q) \}$.

def:

$$M_{\mathcal{F}}(f)(x) = \sup_{S \in \mathcal{F}} \frac{1}{m(S)} \int_S |f(x-y)|\,dy$$

$$\left(\le \sup_{r} \frac{1}{c\, m(B)} \int_B |f(x-y)|\,dy \right)$$

$$M_{\mathcal{F}}(f)(x) \le \frac{1}{c} Mf(x).$$

$\therefore$ all conclusions of thm 1 hold for $M_{\mathcal{F}}(x)$.

$$\therefore \quad \lim_{\substack{S \in \mathcal{F} \\ m(S) \to 0}} \frac{1}{m(S)} \int_S f(x-y)\,dy = f(x)$$

a.e x.

The exceptional set of measure zero may depend on x.

Is there a set of differentiability which works for every regular family $\mathcal{F}$?

(Yes), its called the Lebesgue set of f.

$$\operatorname{Leb}(f) = \left\{ x : \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0 \right\}$$

<u>Claim:</u> $m\left[(\operatorname{Leb}(f))^c \right] = 0$.

<u>Pf:</u> by the diff. thm we have,

For any constant c,

$$(*) \quad \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - c|\, dy = |f(x) - c|.$$

a.e. x.

let $E_c = \{ x : (*) \text{ ~~holds~~ fails.} \}$.

then, $m(E_c) = 0$.

let $c_1, \dots c_n, \dots$ enumerate $\mathbb{Q}$.

then if $x \notin \bigcup_n E_{c_n} \equiv E$.

then $(*)$ holds for every rational c.

$\therefore$ for every real c.

So if $x \in E^c$ then.

$$(**). \quad \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

Now,

$$\left| \frac{1}{m(S)} \int_S f(x-y) \, dy - f(x) \right|$$

$$= \left| \frac{1}{m(S)} \int_S (f(x-y) - f(x)) \, dy \right|.$$

$$\leq \frac{1}{m(S)} \int_S |f(x-y) - f(x)| \, dy$$

$$\leq \frac{1}{c\, m(B(x,r))} \int_{B(x,r)} |f(\cancel{x-y}\, y) - f(x)| \, dy.$$

(some r depending on S).

$\therefore$ f is diff w.r.t. the family $\mathcal{F}$ at each point of $\mathrm{Leb}(f)$.

If $\quad f = \chi_E \qquad E$ msble.

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} \chi_E(y)\,dy = \frac{m(E \cap B(x,r))}{m(B(x,r))}.$$

$$\xrightarrow[r\to 0]{} 1 \quad \text{a.e. } x \in E.$$

$$\xrightarrow[r\to 0]{} 0 \quad \text{a.e. } x \in E^c.$$

Points x s.t.

$$\lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 1.$$

are called

points of density of E.

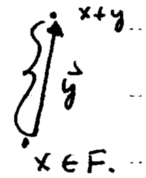
Let F denote a closed subset of $\mathbb{R}^N$.

$$\delta(x) = \delta(x, F) = \inf \{|x-y| : y \in F\}.$$

(why use closed sets?).

$\delta(x) = 0$ iff $x \in F$

if $x \in F$ then $\delta(x+y) \le |y|$



but this can be improved.

In fact,

Prop: F closed.

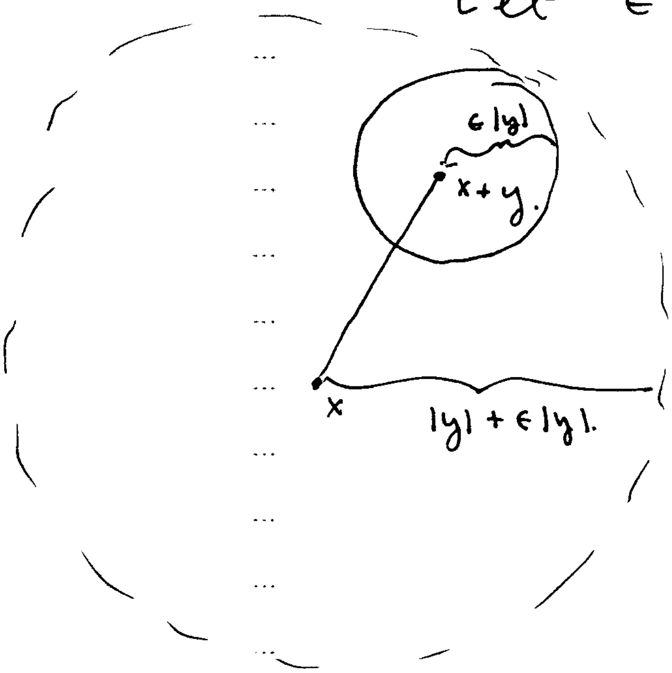
For a.e. $x \in F$

$$\delta(x+y) = o(|y|).$$

i.e. $$\frac{\delta(x+y)}{|y|} \to 0 \qquad |y| \to 0.$$

Pf: Let $x \in F$ be a point of density.

Let $\epsilon > 0$ be given.



If $|y|$ is suff. small. there must be a point of F in the ball

$$B(x+y, \epsilon|y|).$$

If not. then

$$m(F \cap B(x, |y|+\epsilon|y|)) \le m(B(x, |y|+\epsilon|y|)) - m(B(x+y, \epsilon|y|))$$

$$\leq \quad c_n (|y| + \epsilon|y|)^n - c_n (\epsilon|y|)^n$$

$$= c_n \left((1+\epsilon)^n - \epsilon^n \right) |y|^n$$

So $$\frac{m(F \cap B(x, |y| + \epsilon|y|))}{m(B(x, |y| + \epsilon|y|))} \leq \frac{c_n ((1+\epsilon)^n - \epsilon^n) |y|^n}{c_n (1+\epsilon)^n |y|^n}$$

$$= 1 - \left(\frac{\epsilon}{1+\epsilon}\right)^n < 1.$$

So ~~if~~ if $|y|$ is suff. small, then

$\exists\ z \in F$ s.t. $z \in B(x+y, \epsilon|y|)$.

$\therefore \ \delta(x+y, F) \leq \epsilon |y|$.

As $\epsilon > 0$ is arb. we have

$$\delta(x+y) = o(|y|).$$

More is true.

(Marcinkiewicz Integrals).

Consider

$$I(x) = \int_{|y|\leq 1} \frac{\delta(x+y)}{|y|^{n+1}}\, dy.$$

In general, the estimate $\delta(x+y) = o(|y|$ is not enough to make the integral converge.

e.g. $$\int_0^1 \frac{o(r)}{r^2}\, dr$$

might be $$\int_0^1 \frac{\left(\frac{r}{\log 1/r}\right)}{r^2}\, dr = \int_0^1 \frac{1}{r \log 1/r}\, dr$$

$u = \log 1/r$

$du = -\frac{1}{r}\, dr.$

$$= -\int_\infty^0 \frac{1}{u}\, du = +\infty.$$

Let F be closed in $\mathbb{R}^n$.

~~S(x)~~ $\delta(x, F)$.

<u>Thm:</u> With $I(x) = \int_{|y| \leq 1} \frac{\delta(x+y)}{|y|^{n+1}} \, dy$.

a) $x \in F^c \Rightarrow I(x) = +\infty$.

b) ~~$\forall x \in$~~ for a.e. $x \in F$

$I(x) < +\infty$.

<u>Pf:</u> a) is clear since F^c is open. ($\delta(x+y) \geq \epsilon > 0$ For $|y|$ suff small).

b).

Suppose 1st that $m(F^c) < +\infty$.

and let

$$I^*(x) = \int_{\mathbb{R}^n} \frac{\delta(x+y)}{|y|^{n+1}} \, dy.$$

Now consider

$$\int_F I^*(x)\,dx = \int_F \int_{\mathbb{R}^n} \frac{\delta(x+y)}{|y|^{n+1}}\,dy\,dx$$

$$= \int_F \int_{\mathbb{R}^n} \frac{\delta(y)}{|x-y|^{n+1}}\,dy\,dx$$

$$= \int_F \int_{\substack{\cancel{\mathbb{R}^n\setminus F}\\ F^c}} \frac{\delta(y)}{|x-y|^{n+1}}\,dy\,dx$$

$$= \int_{F^c} \left(\int_F \frac{1}{|x-y|^{n+1}}\,dx \right) \delta(y)\,dy.$$

For $\displaystyle\int_F \frac{1}{|x-y|^{n+1}}\,dx$ we have.

$$|x-y| \geq \delta(y) \quad \text{since } x \in F.$$

so $$\int_F \frac{1}{|x-y|^{n+1}}\,dx \leq \int_{|x| \geq \delta(y)} \frac{dx}{|x|^{n+1}} \leq C \int_{r \geq \delta(y)} \frac{dr}{r^2}$$

$$\leq \frac{C}{\delta(y)}.$$

$$\therefore \int_F I^*(x)\,dx \le \int_{F^c} \frac{c}{\delta(y)}\,\delta(y)\,dy = c\,m(F^c).$$

and $I^*(x)$ is $\therefore$ finite for a.e $x \in F$.

$$\therefore \int_{|y|\le 1} \frac{\delta(x+y)}{|y|^{n+1}}\,dy < +\infty \quad \text{for a.e } x\in F.$$

To remove the assumption that $m(F^c) < +\infty$:

let F be closed

let $F_m = F \cup B(0,m)^c$

where $(B(0,m))$ is open.

then F_m is closed and $m(F_m^c) \le m(B(0,m))$

Let $\delta_m(x) = \text{dist}(x, F_m)$.

$\delta(x) = \text{dist}(x, F)$.

if $|x| \le m-2$ and $|y| \le 1$. then
and $x \in F$

$$\delta(x+y) = \delta_m(x+y).$$

So
$$\int_{|y|\le 1} \frac{\delta_m(x+y)}{|y|^{n+1}}\, dx = \int_{|y|\le 1} \frac{\delta(x+y)}{|y|^{n+1}}\, dx$$

$$\forall\, x \in B_{m-2} \cap F.$$

$$\therefore \int_{|y|\le 1} \frac{\delta(x+y)}{|y|^{n+1}}\, dx < +\infty.$$

$\forall$ a.e. $x \in F$.

(letting $m \to \infty$.)