Math 137 11-17-08.

Lebesgue's Differentiation theorem

$f \in L^1_{loc}(\mathbb{R}^N)$ $\left(\int_B |f| dx < +\infty \right)$ for any ball B.

$B(x,r) =$ ball of radius r centered at x.

$m(B(x,r)) =$ Lebesgue measure ($m(E)$ for any set E).

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy = f(x)$$

for a.e x.

The Hardy-Littlewood Maximal Function

for $f \in L^1_{loc}(\mathbb{R}^N)$

is $$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f|\, dy.$$

<u>exercise</u>: Sets of the form $\{x : Mf(x) > \alpha\}$ are open.

Given a fcn g defined on $\mathbb{R}^N$, let

$$\omega(\alpha) =' m\{x : |g(x)| > \alpha\}.$$

From last time we have

(if $\int_{\mathbb{R}^N} |g|^p dx < +\infty$).

$$\int_{\mathbb{R}^N} |g(y)|^p dy = -\int_0^\infty \alpha^p \, d\omega(\alpha)$$

and if $g \in L^\infty$

$$\|g\|_\infty = \inf\{\alpha : \lambda(\alpha) = 0\}.$$

<u>Recall:</u> (Markov, Tchebyshev)

$$\omega(\alpha) \leq \frac{1}{\alpha} \int_{\mathbb{R}^N} |g(y)| \, dy.$$

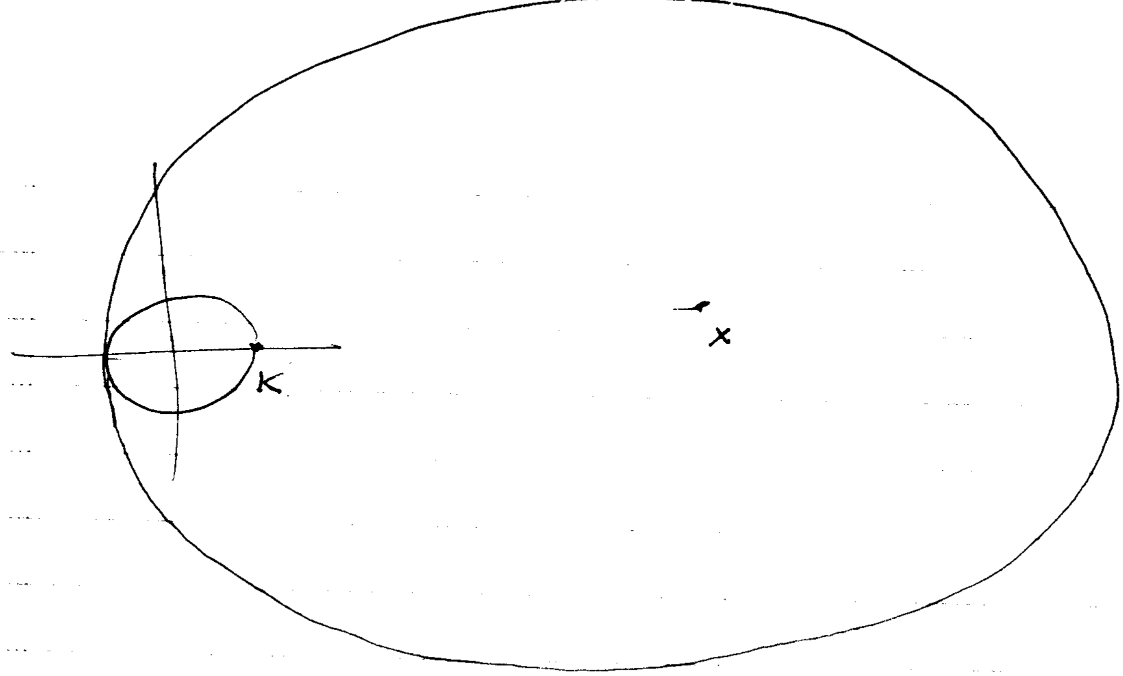
Notice also that if $\exists \underset{\delta > 0}{\epsilon > 0}$ s.t $m\{|f| > \epsilon\} > \delta > 0$.

then $\exists K$ s.t $m\{|x| \leq K, |f| > \epsilon\} > \delta/2 > 0$.

~~then for large~~

So for large x ~~and suff large R~~

$$\cancel{\frac{1}{m(B(x,R))} \int_{B(x,R)} |f| dx \gtrsim \frac{C\epsilon \cdot \delta/2}{R^n}}$$



$$|x|_1 \le |x| + K \le 2|x|.$$

$$M_f(x) \ge \frac{1}{m(B(x, 2|x|))} \int_{B(x,2|x|)} |f|\,dx \ge \frac{C\,\epsilon \cdot \delta/2}{|x|^n}.$$

so $Mf(x)$ is never integrable on $\mathbb{R}^n$.

<u>Theorem</u> f is a function def. on $\mathbb{R}^N$.

a) $f \in L^p(\mathbb{R}^N)$, for some p, $1 \le p \le \infty \Rightarrow Mf < +\infty$ a.e.

b) $f \in L^1(\mathbb{R}^N) \Rightarrow$
$\forall \alpha > 0$

$$m\{x : Mf(x) > \alpha\} \le \frac{A}{\alpha} \int_{\mathbb{R}^N} |f|\,dx.$$

$(A = 5^N$ e.g. $)$
A depends only on dimension

c.) $f \in L^p(\mathbb{R}^N)$ $\quad 1 < p \leq \infty$

$\Rightarrow Mf \in L^p(\mathbb{R}^N)$. and

$\|Mf\|_p \leq A_p \|f\|_p$.

A_p depends only on p and N.

<u>Corollary</u>. <u>Lebesgue's differentiation theorem</u>.

<u>exercise</u>: if $f \in L^p(\mathbb{R}^N)$ some $1 \leq p \leq \infty$
then f is locally integrable.

<u>Remark</u>: Notice the difference in the conclusions
of b) for L^1 and c) for L^p, $p > 1$.

Pf: (theorem).

take $E_\alpha = \{x : Mf(x) > \alpha\}$.

Given $x \in E_\alpha$ $\exists$ $r>0$ s.t.

$$\int_{B(x,r)} |f(y)|dy > \alpha \, m(B(x,r)).$$

The collection of such balls covers E_α

So we can extract a sequence of balls

$\{B_n\}_{n=0}^{\infty}$ s.t.

$$\sum_{n=0}^{\infty} m(B_n) \geq C m(E_\alpha).$$

We then have.

$$\sum_{n=0}^{\infty} \int_{B(x_n,r_n)} |f(y)|dy = \int_{\bigcup_{k=0}^{\infty} B_n} |f(y)|dy \geq \alpha \sum_{n=0}^{\infty} m(B_n)$$

$$\geq C\alpha \, m(E_\alpha).$$

$$\therefore \quad C\alpha \, m(E_\alpha) \leq \int_{\mathbb{R}^N} |f(y)|dy.$$

i.e. we have b) and a) for $p=1$.

Clearly. $\|Mf\|_\infty \le \|f\|_\infty$.

So we consider c) with $1 < p < \infty$.

let

$$f_1(x) = \begin{cases} f(x) & |f(x)| \ge \alpha/2 \\ 0 & \text{else.} \end{cases}$$

then

$$|f(x)| \le |f_1(x)| + \alpha/2.$$

and $|Mf(x)| \le Mf_1(x) + \alpha/2.$

So $\{x : Mf(x) > \alpha\} \subset \{x : Mf_1(x) > \alpha/2\}$

If $f \in L^p$ $1 < p < \infty$. then

$$|f_1 \cdot \tfrac{2}{\alpha}| \ge 1 \qquad \text{when } |f(x)| \ge \alpha/2.$$

so $$\int |f_1| \cdot \tfrac{2}{\alpha}\, dx \le \int \left(|f_1| \tfrac{2}{\alpha}\right)^p dx \le \left(\tfrac{2}{\alpha}\right)^p \int |f|^p dx$$

and $|f_1| \in L^1$.

$$\therefore\ m\{Mf_1(x) > \alpha/2\} \leq \frac{2A}{\alpha}\|f_1\|_1.$$

$$\leq \frac{2A}{\alpha}\int_{|f|>\alpha/2} |f|\,dx.$$

$$\boxed{\text{so that } m\{Mf(x) > \alpha\} \leq \frac{2A}{\alpha}\int_{|f|>\alpha/2} |f|\,dx}$$

let $\omega(\alpha) = m\{M(f) > \alpha\}$.

then

$$\int_{\mathbb{R}^N} (Mf)^p\,dx = -\int_0^\infty \alpha^p\,d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha.$$

$$\leq p\int_0^\infty \alpha^{p-1}\left(\frac{2A}{\alpha}\int_{|f|>\alpha/2} |f(x)|\,dx\right)d\alpha.$$

$$= A\cdot\int_{\mathbb{R}^n} |f(x)|\left(\int_0^{2|f(x)|} \alpha^{p-2}\,d\alpha\right)dx$$

$$= A\int_{\mathbb{R}^N} |f(x)|\cdot\frac{2^{p}}{p-1}|f(x)|^{p-1}\,dx$$

$$= \frac{2^{p}A}{p-1}\int_{\mathbb{R}^N} |f(x)|^p\,dx.$$ (which moves c))

Pf: (Corollary). We may assume that $f \in L^1(\mathbb{R}^n)$.

Let $f_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy$

$r > 0$.

then

Claim: $\| f_r(x) - f(x) \|_{L^1} \to 0 \quad r \to 0$.

Given the claim.

$\exists \; r_n \searrow 0$ s.t. $f_{r_n}(x) \to f(x)$ a.e.

So we just need to show that

$\lim_{r \to 0} f_r(x)$ exists a.e.

For any $g \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ let.

$$Osc(g)(x) = \left| \overline{\lim_{r \to 0}}\, g_r(x) - \underline{\lim_{r \to 0}}\, g_r(x) \right|.$$

If g ~~is cont~~ $\in C^c(\mathbb{R}^N)$ then

$g_r \to g$ unif. and $Osc(g) = 0$.

If $g \in L^1(\mathbb{R}^N)$ then.

$$Osc(g)(x) \le 2Mg(x)$$

So $m\{x : Osc\, g(x) > \epsilon\}$

$$\le m\{x : 2Mg(x) > \epsilon\} \le \frac{2A}{\epsilon}\|g\|_1.$$

(by b) of thm).

~~Let~~ If ~~f~~ ~~be~~ is

Any function in $L^1(\mathbb{R}^N)$ and $\delta > 0$

is given, $\exists$ $g \in C^c(\mathbb{R}^N)$ s.t.

$$\|f - g\|_{L^1(\mathbb{R}^N)} < \delta.$$

then

$$Osc(f) \le Osc(g) + Osc(f-g)$$
$$\le Osc(f-g).$$

So

$$m\{\operatorname{Osc} f > \epsilon\} \leq \cancel{\epsilon} \cdot \left(\frac{2A}{\epsilon}\right)\delta.$$

and as $\delta > 0$ is arbitrary.

$$\operatorname{Osc} f = 0 \quad \text{a.e.}$$

$f \in L^1(\mathbb{R}^N)$.

$\| f(x-y) - f(x) \|_{L^1(\mathbb{R}^N, x)} \to 0$ as $y \to 0$.

if $f \in C^c(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} | f(x-y) - f(x) | \, dx \to 0 \quad y \to 0.$$

by bdd. conv. thm.

given $\epsilon > 0$

let $g \in C^c(\mathbb{R}^N)$ s.t.

$\| f - g \|_{L^1} < \epsilon/4$.

$h = f - g$.

$$\| f(x-y) - f(x) \|_{L^1} \le \underbrace{\| g(x-y) - g(x) \|}_{< \epsilon/2 \text{ if } y \text{ suff smaller.}} + \| h(x-y) - h(x) \|$$

$\| h(x-y) - h(x) \| < \epsilon/2$.

$f_r(x) - f(x)$

$$= \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) - f(x) \, dy.$$

$$= \frac{1}{m(B(0,r))} \int_{B(0,r)} \big(f(x+u) - f(x)\big)\, du.$$

$$= \int_{\mathbb{R}^N} \big(f(x+u) - f(x)\big) \frac{\chi_{B(0,r)}(u)}{m(B(0,r))}\, du.$$

$$\int_{\mathbb{R}^N} |f_r(x) - f(x)|\, dx \le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x+u) - f(x)| \frac{\chi_{B(0,r)}(u)}{m(B(0,r))}\, du\, dx$$

$$\le \int_{\mathbb{R}^N} \frac{\chi_{B(0,r)}(u)}{m(B(0,r))} \left[\int_{\mathbb{R}^N} |f(x+u) - f(x)|\, dx \right] du.$$

$$\le \sup_{|u| \le r} \int_{\mathbb{R}^N} |f(x+u) - f(x)|\, dx$$

$$\longrightarrow 0.$$

$$r \to 0.$$