Math 137 11-12-08.

Theorem: $(E, \mathcal{B}, \mu)$ a measure space.
$f : (E, \mathcal{B}) \to [0, +\infty]$ msble

i) • $\omega(t) = \mu(f > t) \in [0,\infty]$ is a right continuous, non-increasing function
$$\omega : (0,\infty) \to [0,\infty]$$
so it is msble on $((0,\infty), \mathcal{B}_{(0,\infty)})$
and has at most countably many discontinuities

ii) • If $\varphi \in C([0,\infty)) \cap C'((0,\infty))$
is non-decreasing with
$$0 = \varphi(0) < \varphi(t) \quad \forall\, t > 0.$$
and $\varphi(\infty) \equiv \lim_{t\to\infty} \varphi(t)$ then

$$(*) \qquad \int_E \varphi \circ f(x)\, d\mu = \int_{(0,\infty)} \varphi'(t)\, \mu(f > t)\, d\lambda_{\mathbb{R}}$$

iii) • Using the previous theorem, it follows that either $\mu(f > \delta) = +\infty$ for some $\delta > 0$ in which case $(*)$ reads, $+\infty = +\infty$ or
$\varphi'(t)\omega(t) \in R.I\,[\delta, r]$ for each $\delta > 0$ and $r > \delta$
$0 < \delta < r < \infty$.

and

$$\int_{E} \varphi\circ f(x)\, d\mu(x) = \lim_{\substack{\delta\searrow 0 \\ r\nearrow\infty}} (R)\int_{\delta}^{r} \varphi'(t)\,\omega(t)\,dt.$$

Pf: i) $\omega(t)$ is clearly right cont. and non-decreas.

let $\delta = \sup\{t\in(0,\infty) : \omega(t)=+\infty\}$

then $\omega(t) = +\infty \quad \forall\ t<\delta$ and

ω has at most countably many discontinuities on (δ,∞).

Now let $h_n(t) = \omega\left(\frac{k+1}{n}\right)$

$\forall\ t\in\left(\frac{k}{n}, \frac{k+1}{n}\right] \quad k\geq 0,\ n\geq 1.$

then $h_n(t) \to \omega(t) \quad \forall t\in(0,\infty)$ as $n\to\infty$

and h_n is mble $((0,\infty), \mathcal{B}_{(0,\infty)}) \quad \forall n\in\mathbb{Z}^+$.

so $\omega(t)$ is mble $((0,\infty), \mathcal{B}_{(0,\infty)})$.

ii) We claim that if $\omega(\delta)=+\infty$ for some $\delta>0$ then both sides of $(*)$ are $+\infty$.

For any $\delta>0$ we have (since φ is non-decr.)

$\int_E \varphi\circ f\, d\mu \geq \varphi(\delta)\,\mu(f>\delta) =$

and by exercise 3.3.19 we have

$$\lim_{\alpha \searrow 0} \int_{(\alpha, \delta]} \varphi'(t)\,dt = \lim_{\alpha \searrow 0} (R)\int_{\alpha}^{\delta} \varphi'(t)\,dt = \varphi(\delta)$$

So $\displaystyle\int_{(0,\infty)} \varphi'(t)\,\omega(t)\,d\lambda \geq \int_0^{\delta} \varphi'(t)\omega(t)\,d\lambda_{\mathbb{R}}$

$$\geq \varphi(\delta)\omega(\delta)$$

So if $\omega(\delta) = +\infty$ for some $\delta > 0$ then both sides of $(*)$ are $+\infty$.

Assume, as we may, that $\omega(\delta) < +\infty \ \forall\, \delta > 0$.

Then $\mu_f \big|_{\mathcal{B}[\delta,\infty)}$ is a finite measure

$\forall\, \delta > 0$.

Given $\delta > 0$ take $\psi_\delta(t) = \mu_f((\delta, t])$
$t \in [\delta, \infty)$

then $\forall\, r \in (\delta, \infty)$,

$$\int_{\{\delta < f \leq r\}} (\varphi \circ f)\,d\mu = \int_{(\delta, r]} \varphi\,d\mu_f \qquad \text{(our result from last time)}$$

$$= (R)\int_{[\delta, r]} \varphi(t)\,d\psi_\delta(t) \qquad \text{(previous thm)}.$$

$$= \varphi(r)\psi_\delta(r) - (R)\int_{[\delta,r]} \psi_\delta(t)\varphi'(t)\,dt.$$

(integration by parts).

$$= \varphi(\delta)\psi_\delta(r) + (R)\int_{[\delta,r]} [\psi_\delta(r) - \psi_\delta(t)]\varphi'(t)\,dt.$$

$$= \varphi(\delta)\mu(\delta < f \le r) + (R)\int_{[\delta,r]} \mu(t < f \le r)\varphi'(t)\,dt$$

$$= \varphi(\delta)\mu(\delta < f \le r) + \int_{[\delta,r]} \mu(t < f \le r)\varphi'(t)\,d\lambda_{\mathbb{R}}$$

(previous thm again).

$\therefore \quad \forall r$

$$\int_{\{\delta < f \le r\}} (\varphi \circ f - \varphi(\delta))\,d\mu = \int_{[\delta,r]} \mu(t < f \le r)\varphi'(t)\,d\lambda_{\mathbb{R}}.$$

and by M.C.T we have. $(r \nearrow \infty)$

a) $$\int_{\{\delta < f < \infty\}} (\varphi \circ f - \varphi(\delta))\,d\mu = \int_{[\delta,\infty)} \varphi'(t)\,\mu(t < f < \infty)\,d\lambda_{\mathbb{R}}.$$

also,

b) $$\int_{\{f=\infty\}} (\varphi\circ f - \varphi(\delta))\,d\mu = [\varphi(\infty) - \varphi(\delta)]\,\mu\{f=\infty\}$$

$$= \int_{(\delta,\infty)} \mu(f=\infty)\,\varphi'(t)\,d\lambda_{\mathbb{R}}(t).$$

(exercise 3.3.19 again).

adding a) and b) we get.

$$\int_{\{f>\delta\}} (\varphi\circ f(x) - \varphi(\delta))\,d\mu = \int_{(\delta,\infty)} \mu(f>t)\,\varphi'(t)\,d\lambda_{\mathbb{R}}(t).$$

Since $(\varphi\circ f(x) - \varphi(\delta))\chi_{\{f>\delta\}} \nearrow$ as $\delta \searrow 0$.

M.C.T $\Longrightarrow$

$$\lim_{\delta\searrow 0} \int_{\{f>\delta\}} (\varphi\circ f(x) - \varphi(\delta))\,d\mu = \int_E \varphi\circ f(x)\,d\mu(x).$$

Finally, we have.

$$\int_{(0,\infty)} \varphi'(t)\,\omega(t)\,d\lambda_{\mathbb{R}}(t) = \lim_{\substack{\delta\searrow 0\\ r\nearrow\infty}} \int_\delta^r \varphi'(t)\,\omega(t)\,d\lambda_{\mathbb{R}}(t)$$

$$= \lim_{\substack{\delta\searrow 0\\ r\nearrow\infty}} (R)\int_\delta^r \varphi'(t)\,\omega(t)\,dt. \blacksquare$$

by prev. thm.

A covering lemma:

$E \subset \mathbb{R}^n$ Lebesgue msble $(\mathbb{R}^n, \overline{\mathcal{B}_{\mathbb{R}^n}}, m)$.

covered by $\bigcup_{j \in J} B_j$

where each B_j is a ball

and $\sup_{j \in J} \operatorname{diam}(B_j) = M < +\infty$.

$\exists$ an at most countable collection

$B_1, B_2, B_3 \ldots B_n \ldots \subset \{B_j\}_{j \in J}$

s.t $B_m \cap B_n = \emptyset$ if $m \neq n$ $m, n \in \mathbb{Z}^+$.

and s.t.

$$\sum_k m(B_k) \geq C\, m(E).$$

where $C > 0$ is a constant depending only on the dimension.

$$\therefore \quad m(E) \leq \sum_{k=1}^{\infty} m(B_k^*) \leq 5^n \sum_{n=1}^{\infty} m(B_n).$$

Choose B_1 s.t.

$$\operatorname{diam} B_1 \geq \frac{1}{2} \sup_{j \in J} \{\operatorname{diam}(B_j)\}.$$

Suppose that $B_1, \dots, B_k$ have been chosen and are disjoint

$$B_m \cap B_n = \emptyset \qquad \begin{array}{l} m \neq n \\ 1 \leq m \leq k \\ 1 \leq n \leq k. \end{array}$$

Choose B_{k+1} disjoint from all $B_1, \dots, B_k$ and s.t.

$$\operatorname{diam} B_{k+1} \geq \frac{1}{2} \sup_{\{j : B_j \text{ is disjoint from all } B_1, \dots, B_k\}} \{\operatorname{diam} B_j\}.$$

If there is no ~~such~~ ball remaining which is disjoint from $B_1, \dots, B_k$.
the process stops and the sequence of balls is finite.

otherwise we continue

~~If~~

and the sequence defined inductively by this procedure may be countably infinite.

If $\sum_{n=1}^{\infty} m(B_n) = +\infty$

then we are done.

So assume that $\sum_{n=1}^{\infty} m(B_n) < +\infty$.

For each k, let $5B_k$ be the ball with same center as B_k and $\mathrm{diam}(5B_k) = 5\,\mathrm{diam}(B_k)$.

Claim: for any $j \in J$

$$B_j \subset \bigcup_{k=1}^{\infty} (5B_k).$$

The claim is clearly if

$$B_j \in \{B_1, B_2 \dots B_k \dots\}$$

so assume B_j was <u>not</u> chosen.

If the collection is Finite

$$\{B_1, \dots, B_n\}$$

then B_j intersects one of the $\{B_1, \dots, B_n\}$ (else we would have added B_j to the collection at step $k+1$).

If the collection is infinite then since $\sum_{n=1}^{\infty} m(B_n) < +\infty$

$$m(B_n) \to 0 .$$

Take the smallest k s.t

$$\operatorname{diam}(B_{k+1}) < \frac{1}{2} \operatorname{diam}(B_j).$$

then B_j must intersect one of $B_1, \ldots, B_k$.

If not then B_j would have been chosen at step $k+1$ over B_{k+1} because

$$\operatorname{diam}(B_j) > 2 \operatorname{diam}(B_{k+1})$$

Since k was the smallest integer s.t.

$$\operatorname{diam}(B_{k+1}) < \frac{1}{2} \operatorname{diam}(B_j)$$

we must have.

$$\operatorname{diam}(B_i) \geq \frac{1}{2} \operatorname{diam}(B_j)$$

$$\forall\, i = 1, 2, \ldots, k.$$

So if $B_{i^*} \cap B_j \neq \emptyset$

(as it must be for some $i^* \in \{1, \ldots, k\}$.

then $B_j \subseteq 5B_{i^*}$.