Math 137 11-10-08.

exercise 1.2.6.

exercise 3.3.19.

(1.2.6) • If ψ is non-decreasing on J (an interval) then

a bounded function φ is R.I-ψ on J iff given $\epsilon > 0$ $\exists\ \delta > 0$ s.t.

$$\sum_{\{I \in \mathcal{C} :\ \underline{osc(\varphi, I)} \geq \epsilon\}} \Delta_I \psi < \epsilon$$

whenever $\mathcal{C}$ is a good cover of J s.t $\|\mathcal{C}\| < \delta$.

$$\boxed{osc(\varphi, I) = \sup_I \varphi - \inf_I \varphi}$$

"good" $\equiv$ $\begin{cases} \text{non-overlapping} \\ \text{finite} \\ \text{exact} \end{cases}$

"δ-good" $\equiv$ $\{$~~good~~, $\|\mathcal{C}\| < \delta\}$.

<u>Pf:</u> If the bounded function φ is R.I-ψ on J then given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $\forall$ ~~good $\mathcal{C}$~~ ~~with~~ δ-good $\mathcal{C}$

$$\sum_{I \in \mathcal{C}} osc(\varphi, I) \Delta_I \psi < \epsilon^2$$

Consequently, for all δ-good $\mathcal{C}$ we have

$$\epsilon \cdot \sum_{\{I \in \mathcal{C} :\ osc(\varphi, I) \geq \epsilon\}} \Delta_I \psi \leq \sum_{I \in \mathcal{C}} osc(\varphi, I) \Delta_I \psi < \epsilon^2$$

Conversely, (we don't need the converse).

$$\sum_{I\in\mathcal{C}} osc(\varphi, I)\,\Delta_I\psi = \sum_{\{I\in\mathcal{C}:\, osc(\varphi,I)\geq\epsilon\}} + \sum_{\{I\in\mathcal{C}:\, osc(\varphi,I)<\epsilon\}}$$

$$\leq (\sup_J \varphi) \sum_{\{I\in\mathcal{C}:\, osc(\varphi,I)\geq\epsilon\}} \Delta_I\psi + \epsilon\,\Delta_J\psi$$

~~[illegible]~~ so the 2nd condition implies that the bdd fcn φ is RI-ψ on J.

<u>3.3.19</u>. $J\subset\mathbb{R}^N$ - a closed rectangle

$f: J\to\mathbb{R}$ continuous

- $$(R)\int_J f(x)\,dx = \int_J f(x)\, d\underbrace{\lambda_{\mathbb{R}^N}(x)}_{\text{Lebesgue measure}}$$

- If $f\in L^1(d\lambda_{\mathbb{R}^N})$ is continuous then

$$\int f(x)\, d\lambda_{\mathbb{R}^N} = \lim_{J\nearrow\mathbb{R}^N} (R)\int_J f(x)\,dx$$

i.e. given $\epsilon>0$ $\exists$ J_ϵ, a rectangle s.t.

$$\left| \int f(x)\, d\lambda_{\mathbb{R}^N}(x) - (R)\int_J f(x)\,dx \right| < \epsilon.$$

Pf: Divide each edge of J in dyadic fashion and consider the corresponding subdivision of J, for each dyadic level n.

If $I \subseteq J$ is a rectangle in the n^{th} subdivision ~~let~~ $\cancel{f_I = \inf_I |f|}$ choose $x_I \in I$ s.t $|f(x_I)| = \inf_I |f|$. and define $f_n \equiv \sum_{\{I \in n^{th} \text{ partition}\}} f(x_I) \chi_I$.

Then $f_n \to f$ pointwise on J

and $|f_n| \le |f|$ $\forall n$

so by L.D.C.T. $\int_J f_n \, d\lambda_{\mathbb{R}^N} \to \int_J f \, d\lambda_{\mathbb{R}^N}$

but $\int_J f_n \, d\lambda_{\mathbb{R}^N}$ is a Riemann Sum for $(R)\int_J f(x)\,dx$.

If $f \in L^1(d\lambda_{\mathbb{R}^N})$ is continuous and $J \subset \mathbb{R}^N$ are rectangles s.t $J \nearrow \mathbb{R}^N$ (and (e.g.) J are centered at the origin.)

then $f\chi_J \to f$ pointwise

and $|f\chi_J| \le |f|$.

So $\int_{\mathbb{R}^N} f \chi_J \, d\lambda_{\mathbb{R}^N} \to \int_{\mathbb{R}^N} f \, d\lambda_{\mathbb{R}^N}$.

and the 2nd conclusion of the exercise follows easily from the 1st.

<u>Theorem:</u> ν a finite measure on $((a,b], \mathcal{B}_{(a,b]})$ $-\infty < a < b < \infty$.

$$\psi(t) = \nu((a,t]) \quad a \le t \le b.$$

$$(\psi(a) = \nu(\emptyset) = 0.).$$

monotonicity of measures

- ψ is right continuous on $[a,b)$ $(\checkmark)$
- ψ is non-decreasing on $[a,b]$ $(\checkmark)$.
- $\psi(a) = 0$ $(\checkmark)$ $\psi(t) - \psi(t-) = \nu\{t\}$ $\forall t \in (a,b]$ $(\checkmark)$ where $\psi(t-) = \lim_{s \nearrow t} \psi(s)$.

- a bdd φ on $[a,b]$ is R.I-ψ on $[a,b]$ iff φ is cont. ν-a.e. on $(a,b]$.

- if φ is R.I-ψ on $[a,b]$ then φ is msble on $((a,b], \overline{\mathcal{B}}^{\nu}_{(a,b]})$ and.

$$\int_{(a,b]} \varphi \, d\bar{\nu} = (R)\int_{[a,b]} \varphi(t)\, d\psi(t).$$

<u>Pf:</u> We take (as we may) ν to be defined on $([a,b], \mathcal{B}_{[a,b]})$ by

$$\nu(\Gamma) \equiv \nu(\Gamma \cap (a,b]) \quad \forall\, \Gamma \in \mathcal{B}_{[a,b]}$$

and check off the 1st 3 conclusions. Note that ψ has at most countably many points of discontinuity since it is non-decreasing on $[a,b]$.

Suppose that the bounded function φ is Riemann integrable $-\psi$ on $[a,b]$.

For each $n \geq 1$ choose a $\frac{1}{n}$-good cover $\mathcal{C}_n$ of $[a,b]$ by intervals I s.t. ψ is continuous at I^- for each $I \in \mathcal{C}_n$.

Let $\Delta = \bigcup_{n=1}^{\infty} \{ I^- : I \in \mathcal{C}_n \}$ and note that $\nu(\Delta) = 0$.

for each $m \geq 1$ let

$$\mathcal{C}_{m,n} = \left\{ I \in \mathcal{C}_n : \operatorname{osc}_I \varphi \geq \frac{1}{m} \right\}.$$

If φ is not continuous at $t \in (a,b]$

then $\exists\, m \in \mathbb{Z}^+$ s.t. if I^* is any interval containing t then

$$\operatorname{osc}_{I^*} \varphi \geq \frac{1}{m}.$$

$$\therefore \quad \left\{ t \in (a,b] \setminus \Delta : \varphi \text{ is not continuous at } t \right\}$$

$$\subseteq \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \Big(\bigcup_{I \in \mathcal{C}_{m,n}} I \Big).$$

Now

$$\bigcap_{n=1}^{\infty} \Big(\bigcup_{I \in \mathcal{C}_{m,n}} I \Big) \subset \bigcup_{I \in \mathcal{C}_{m,n^*}} I \qquad \text{for each } n^* \in \mathbb{Z}^+$$

and $v\Big(\cancel{I} \bigcup_{I \in \mathcal{C}_{m,n^*}} I \Big) = \sum_{I \in \mathcal{C}_{m,n^*}} \Delta_I \psi \longrightarrow 0$

as $n^* \to \infty$

by exercise 1.2.26.

So

$$v\Big(\bigcap_{n=1}^{\infty} \bigcup_{I \in \mathcal{C}_{m,n}} I \Big) \leq \lim_{n \to \infty} \sum_{I \in \mathcal{C}_{m,n}} \Delta_I \psi = 0.$$

$\therefore \quad \nu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left(\bigcup_{I \in \mathcal{C}_{m,n}} I \right) \right) = 0$

and φ is cont. ν-a.e. on $[a,b]$.

Conversely,

if φ is cont. ν-a.e. on $(a,b]$

let for each $n \in \mathbb{Z}^+$ $\mathcal{C}_n$ be a ~~sequence of~~ $\frac{1}{n}$-good cover of (a,b) by intervals I.

For $n \in \mathbb{Z}^+$ let

$$\overline{\varphi_n}(t) = \sup_I \varphi \qquad t \in I \setminus \{I^-\}$$

$$\underline{\varphi_n}(t) = \inf_I \varphi \qquad I \in \mathcal{C}_n.$$

$\overline{\varphi_n}$ and $\underline{\varphi_n}$ are msble on $((a,b], \mathcal{B}_{(a,b]})$

and $\forall n \in \mathbb{Z}^+$

$$\inf_{(a,b]} \varphi \le \underline{\varphi}_n \le \varphi \le \overline{\varphi_n} \le \sup_{(a,b]} \varphi$$

Since φ is cont ν-a.e.

$$\varphi = \lim_{n\to\infty} \underline{\varphi}_n = \lim_{n\to\infty} \overline{\varphi_n} \quad \text{a.e.-}\nu.$$

(φ is continuous by assumption at ν almost every point I^- $I \in \mathcal{C}_n$.)

$\therefore$ φ is msble on $\left((a,b], \bar{\mathcal{B}}^{\nu}_{(a,b]}\right)$.

and (L.D.C.T).

$$\lim_{n\to\infty} \int_{(a,b]} \underline{\varphi}_n \, d\nu = \int_{(a,b]} \varphi \, d\bar{\nu} = \lim_{n\to\infty} \int_{(a,b]} \bar{\varphi}_n \, d\nu$$

Consequently,

$$\sum_{I \in \mathcal{C}_n} (\sup_I \varphi - \inf_I \varphi) \Delta_I \psi = \int_{(a,b]} (\bar{\varphi}_n - \underline{\varphi}_n) \, d\nu \to 0.$$

as $n \to \infty$

so $\varphi \in R.I-\psi\ [a,b]$

and $$\int_{(a,b]} \varphi \, d\bar{\nu} = (R) \int_{[a,b]} \varphi(t) \, d\psi(t).$$

$\blacksquare$.

<u>Theorem:</u> $(E, \mathcal{B}, \mu)$ a measure space.

$f: (E, \mathcal{B}) \to [0, +\infty]$ msble

i) • $\omega(t) = \mu(f > t) \in [0, \infty]$ is a right continuous, non-increasing function

$\omega: (0, \infty) \to [0, \infty]$

so it is msble on $((0,\infty), \mathcal{B}_{(0,\infty)})$

and has at most countably many discontinuities

ii) • If $\varphi \in C([0,\infty)) \cap C'((0,\infty))$

is non-decreasing with

$0 = \varphi(0) < \varphi(t) \quad \forall \ t > 0.$

and $\varphi(\infty) \equiv \lim\limits_{t \to \infty} \varphi(t)$ then

$$(*) \qquad \int_E \varphi \circ f(x) \, d\mu = \int_{(0,\infty)} \varphi'(t) \mu(f > t) \, d\lambda_{\mathbb{R}}$$

iii) • Using the previous theorem, it follows that either $\mu(f > \delta) = +\infty$ for some $\delta > 0$ in which case $(*)$ reads $+\infty = +\infty$ or

$\varphi'(t)\omega(t) \in R.I[\delta, r]$ for each $\delta > 0$ and $r > \delta$ $0 < \delta < r < \infty$.

and

$$\int_E \varphi\circ f(x)\,d\mu(x) = \lim_{\substack{\delta\searrow 0\\ r\nearrow\infty}} (R)\int_\delta^r \varphi'(t)\,\omega(t)\,dt.$$

Pf: i) $\omega(t)$ is clearly right cont. and non-decreas.

let $\delta = \sup\{t\in(0,\infty) : \omega(t)=+\infty\}$

then $\omega(t)=+\infty \quad \forall\ t<\delta$ and

ω has at most countably many discontinuities on (δ,∞).

Now let $h_n(t) = \omega\left(\frac{k+1}{n}\right)$

$\forall\ t\in\left(\frac{k}{n}, \frac{k+1}{n}\right] \qquad k\geq 0,\ n\geq 1.$

then $h_n(t)\to\omega(t) \quad \forall t\in(0,\infty)$ as $n\to\infty$

and h_n is mble $\left((0,\infty),\mathcal{B}_{(0,\infty)}\right)$ $\forall n\in\mathbb{Z}^+$.

so $\omega(t)$ is mble $\left((0,\infty),\mathcal{B}_{(0,\infty)}\right)$.

ii) We claim that if $\omega(\delta)=+\infty$ for some $\delta>0$

then both sides of $(*)$ are $+\infty$.

For any $\delta>0$ we have (since φ is non-decr.)

$$\int_E \varphi\circ f\,d\mu \geq \varphi(\delta)\,\mu(f>\delta) =$$

and by exercise 3.3.19 we have

$$\lim_{\alpha \searrow 0} \int_{(\alpha,\delta]} \varphi'(t)\,dt = \lim_{\alpha \searrow 0} (R)\int_{\alpha}^{\delta} \varphi'(t)\,dt = \varphi(\delta)$$

So
$$\int_{(0,\infty)} \varphi'(t)\,\omega(t)\,d\lambda \geq \int_0^{\delta} \varphi'(t)\,\omega(t)\,d\lambda_{\mathbb{R}}$$
$$\geq \varphi(\delta)\,\omega(\delta)$$

So if $\omega(\delta) = +\infty$ for some $\delta > 0$ then both sides of $(\#)$ are $+\infty$.

Assume, as we may, that $\omega(\delta) < +\infty \ \forall\, \delta > 0$.

Then $\mu_f \big|_{\mathcal{B}[\delta,\infty)}$ is a finite measure

$\forall\, \delta > 0$.

Given $\delta > 0$ take $\psi_\delta(t) = \mu_f((\delta, t])$
$t \in [\delta, \infty)$

then $\forall\, r \in (\delta, \infty)$,

$$\int_{\{\delta < f \leq r\}} (\varphi \circ f)\,d\mu = \int_{(\delta, r]} \varphi\,d\mu_f \qquad \text{(our result from last time)}$$
$$= (R)\int_{[\delta, r]} \varphi(t)\,d\psi_\delta(t) \qquad \text{(previous thm)}.$$

$$= \varphi(r)\,\psi_\delta(r) - (R)\int_{[\delta,r]} \psi_\delta(t)\,\varphi'(t)\,dt.$$

(integration by parts).

$$= \varphi(\delta)\,\psi_\delta(r) + (R)\int_{[\delta,r]} [\psi_\delta(r) - \psi_\delta(t)]\,\varphi'(t)\,dt.$$

$$= \varphi(\delta)\,\mu(\delta < f \leq r) + (R)\int_{[\delta,r]} \mu(t < f \leq r)\,\varphi'(t)\,dt$$

$$= \varphi(\delta)\,\mu(\delta < f \leq r) + \int_{[\delta,r]} \mu(t < f \leq r)\,\varphi'(t)\,d\lambda_{\mathbb{R}}$$

(previous thm again).

$\therefore \quad \forall r$

$$\int_{\{\delta < f \leq r\}} (\varphi\circ f - \varphi(\delta))\,d\mu = \int_{[\delta,r]} \mu(t < f \leq r)\,\varphi'(t)\,d\lambda_{\mathbb{R}}.$$

and by M.C.T we have. $(r \nearrow \infty)$

a) $$\int_{\{\delta < f < \infty\}} (\varphi\circ f - \varphi(\delta))\,d\mu = \int_{[\delta,\infty)} \varphi'(t)\,\mu(t < f < \infty)\,d\lambda_{\mathbb{R}}.$$

also,

b) $$\int_{\{f=\infty\}} (\varphi\circ f - \varphi(\delta))\,d\mu = [\varphi(\infty) - \varphi(\delta)]\,\mu\{f=\infty\}$$

$$= \int_{(\delta,\infty)} \mu(f=\infty)\,\varphi'(t)\,d\lambda_{\mathbb{R}}(t).$$

(exercise 3.3.19 again).

adding a) and b) we get.

$$\int_{\{f>\delta\}} (\varphi\circ f(x) - \varphi(\delta))\,d\mu = \int_{(\delta,\infty)} \mu(f>t)\,\varphi'(t)\,d\lambda_{\mathbb{R}}(t).$$

Since $(\varphi\circ f(x) - \varphi(\delta))\chi_{\{f>\delta\}} \nearrow$ as $\delta \searrow 0$.

M.C.T $\Longrightarrow$

$$\lim_{\delta\searrow 0} \int_{\{f>\delta\}} (\varphi\circ f(x) - \varphi(\delta))\,d\mu = \int_E \varphi\circ f(x)\,d\mu(x).$$

Finally, we have.

$$\int_{(0,\infty)} \varphi'(t)\,\omega(t)\,d\lambda_{\mathbb{R}}(t) = \lim_{\substack{\delta\searrow 0\\ r\nearrow\infty}} \int_\delta^r \varphi'(t)\,\omega(t)\,d\lambda_{\mathbb{R}}(t)$$

$$= \lim_{\substack{\delta\searrow 0\\ r\nearrow\infty}} (R)\int_\delta^r \varphi'(t)\,\omega(t)\,dt.$$

by prev. thm