Math 137 11-5-08.

Last time

(Tonelli).

$(E_1, \mathcal{B}_1, \mu_1) \quad (E_2, \mathcal{B}_2, \mu_2)$

$f : (E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2) \to [0, \infty]$ mble.

$\exists\ \nu$ on $\mathcal{B}_1 \times \mathcal{B}_2$ a measure

s.t. $\nu(\Gamma_1 \times \Gamma_2) = \mu_1(\Gamma_1)\mu_2(\Gamma_2)$

$\forall\ \Gamma_i \in \mathcal{B}_i \quad i = 1, 2.$

and $\forall$ $f(x_1, x_2)$ as above. $(f \geq 0)$.

$$\int_{E_1 \times E_2} f \, d\nu = \int_{E_2} \left( \int_{E_1} f \, d\mu_1 \right) d\mu_2$$

$$= \int_{E_1} \left( \int_{E_2} f \, d\mu_2 \right) d\mu_1$$

Theorem: (Fubini).

$f : E_1 \times E_2 \to \overline{\mathbb{R}}$ mble. $\mathcal{B}_1 \times \mathcal{B}_2$.

$f \in L^1(\mu_1 \times \mu_2)$ iff

$$\int_{E_1} \left( \int_{E_2} |f| \, d\mu_2 \right) d\mu_1 < +\infty.$$

iff $\int_{E_2} \left( \int_{E_1} |f| \, d\mu_1 \right) d\mu_2 < +\infty.$

With

$$\Lambda_1 = \{ x_1 \in E_1 : \quad f(x_1, \cdot) \in L^1(\mu_2) \}$$

$$\Lambda_2 = \{ x_2 \in E_2 : \quad f(\cdot, x_2) \in L^1(\mu_1) \}.$$

The functions

$$f_1(x_1) = \begin{cases} \int_{E_2} f(x_1, x_2)\, d\mu_2(x_2) & x_1 \in \Lambda_1 \\ 0 & \text{else} \end{cases}$$

$$f_2(x_2) = \begin{cases} \int_{E_1} f(x_1, x_2)\, d\mu_1(x_1) & x_2 \in \Lambda_2 \\ 0 & \text{else} \end{cases}$$

are $\mathbb{R}$ valued msble on $(E_i, \mathcal{B}_i)$.

If $f$ is $\mu_1 \times \mu_2$ integrable

then $\mu_i(\Lambda_i^c) = 0$, $f_i \in L^1(\mu_i)$

and

$$(*) \quad \int_{E_i} f_i(x_i)\, d\mu_i(x_i) = \int_{E_1 \times E_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2)$$

$i = 1, 2.$

Pf: (last statement;[*] the others are all easy consequences of Tonelli).

$$\int_{E_1 \times E_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2) = \int_{\Lambda_1 \times E_2} f \, d(\mu_1 \times \mu_2)$$

$$= \int_{\Lambda_1 \times E_2} f^+ \, d(\mu_1 \times \mu_2) - \int_{\Lambda_1 \times E_2} f^- \, d(\mu_1 \times \mu_2)$$

$$= \int_{\Lambda_1} \left( \int_{E_2} f^+ \, d\mu_2 \right) d\mu_1 - \int_{\Lambda_1} \left( \int_{E_2} f^- \, d\mu_2 \right) d\mu_1$$

$$= \int_{\Lambda_1} f_1(x_1)\, d\mu_1$$

and similarly with $1 \leftrightarrow 2$.

## Changes of Variable

$(E_1, \mathcal{B}_1)$ ~~msbld~~ $(E_2, \mathcal{B}_2)$ msble spaces.

$\mu$ a measure on $(E_1, \mathcal{B}_1)$

$\Phi : (E_1, \mathcal{B}_1) \to (E_2, \mathcal{B}_2)$ a msble map.

$$\boxed{\begin{array}{l} \text{i.e. if } \Gamma_2 \in \mathcal{B}_2 \text{ then} \\ \Phi^{-1}(\Gamma_2) = \{ x \in E_1 : \Phi(x) \in \Gamma_2 \} \in \mathcal{B}_1. \end{array}}$$

Define the measure $\Phi_* \mu$ on $(E_2, \mathcal{B}_2)$

by $\quad (\Phi_* \mu)(\Gamma) = \mu(\Phi^{-1}(\Gamma))$

for $\Gamma \in \mathcal{B}_2$.

$$\Big( \Phi_* \mu\Big(\bigcup_1^\infty \Gamma_n\Big) = \mu\Big(\phi^{-1}\Big(\bigcup_1^\infty \Gamma_n\Big)\Big)$$

$$= \mu\Big(\bigcup_1^\infty \phi^{-1}(\Gamma_n)\Big)$$

$$= \sum_1^\infty \mu(\phi^{-1}(\Gamma_n)) = \sum_1^\infty \phi_* \mu(\Gamma_n)$$

if $\Gamma_n \cap \Gamma_m = \emptyset \quad m \neq n \Big)\Big).$

Lemma: for every $\varphi \geq 0$ msble on $(E_2, \mathcal{B}_2)$

$$(*) \quad \int_{E_2} \varphi \, d(\Phi_* \mu) = \int_{E_1} \varphi \circ \Phi \, d\mu$$

$\varphi \in L^1(E_2, \mathcal{B}_2, \Phi_* \mu)$ iff $\varphi \circ \Phi \in L^1(E_1, \mathcal{B}_1, \mu)$

and $(*)$ holds $\forall \varphi \in L^1(E_2, \mathcal{B}_2, \Phi_* \mu)$.

Pf: The last two statements follow by considering positive and negative parts separately.

If $\varphi = \chi_\Gamma$ for $\Gamma \in \mathcal{B}_2$

$$\int_{E_2} \chi_\Gamma \, d(\Phi_* \mu) = \Phi_* \mu(\Gamma) = \mu(\Phi^{-1}(\Gamma))$$

$$= \int_{E_1} \chi_{\Phi^{-1}(\Gamma)} \, d\mu_1$$

$$= \int_{E_1} \chi_\Gamma(\Phi(x)) \, d\mu_1(x).$$

By linearity $(*)$ holds when $\varphi$ is a non-negative msble simple function on $(E_2, B_2)$.

By M.C.T. $(*)$ holds for all $\varphi \geq 0$ msble on $(E_2, B_2)$.

---

If the mapping above is

$$f : (E, \mathcal{B}, \mu) \longrightarrow (\bar{\mathbb{R}}, \cancel{B_{\bar{\mathbb{R}}})}$$

we take $\mu_f = f_* \mu$ on $(\bar{\mathbb{R}}, B_{\bar{\mathbb{R}}})$

then if $\varphi \geq 0$ is msble on $(\bar{\mathbb{R}}, B_{\bar{\mathbb{R}}})$ we have.

$$\int_E (\varphi \circ f)(x)\, d\mu(x) = \int_{\bar{\mathbb{R}}} \varphi(t)\, d\mu_f(t).$$

The L.H.S. can often be evaluated as a Riemann integral or limit of Riemann integrals.

<u>Theorem</u>: $\nu$ a finite measure on

$$((a,b], \mathcal{B}_{(a,b]})$$

$$-\infty < a < b < \infty$$

$$\psi(t) = \nu((a,t]) \qquad t \in [a,b]$$

$$(\psi(a) = \nu(\emptyset) = 0).$$

then

$\psi$ is right-continuous on $[a,b)$

non-decreasing on $[a,b]$.

for each $t \in (a,b]$

$$\psi(t) - \psi(t-) = \nu(\{t\}).$$

$$\left(\psi(t-) = \lim_{s \nearrow t} \psi(s).\right)$$

If $\varphi$ is a bdd fcn on $[a,b]$, then

$\varphi$ is $(R.I)(\psi)$ on $[a,b]$ iff

$\varphi$ is continuous a.e. $(\bar{\nu})$ on

$(a,b]$

in which case $\varphi$ is msble $\left((a,b], \overline{\mathcal{B}}^{\bar{\nu}}_{(a,b]}\right)$

$$\int_{(a,b]} \varphi \, d\bar{\nu}$$

$$= (R) \int_{[a,b]} \varphi(t) \, d\psi(t).$$

( Conversely, each right cont non-decreasing $\psi$ gives a (Borel) measure $\mu$. We'll see this later )

<u>Pf:</u> Consider $\nu$ to be defined

on $\left([a,b], \mathcal{B}_{[a,b]}\right)$

$\nu(\Gamma) = \nu(\Gamma \cap (a,b]) \quad \Gamma \in \mathcal{B}_{[a,b]}$

$\Psi$ is non decreasing ✓.

so has at most countably many points of discontinuity.

$\Psi(a) = \nu(\emptyset) = 0$.

If $s \in [a,b)$

$$\Psi(s) = \nu((a,s]) = \lim_{t \searrow s} \nu((a,t]) = \lim_{t \searrow s} \Psi(t).$$

so $\Psi$ is right continuous

for each $t \in (a,b]$.

$$\Psi(t) - \Psi(t-) = \nu((a,t]) - \lim_{s \nearrow t} \nu((a,s]) = \lim_{s \nearrow t} \nu((s,t]). = \nu(\{t\}).$$

Suppose that $\varphi$ is $RI$-$(\psi)$ on $[a,b]$.

(Claim: $\varphi$ is cont a.e. $\nu$ on $(a,b]$).

For each $n \geq 1$, choose a finite, non-overlapping exact cover of $[a,b]$ $(\mathcal{C}_n)$, by intervals $I$ such that

$\|\mathcal{C}_n\| < \frac{1}{n}$ and s.t $\psi$ is cont at $I^-$ $\forall\, I \in \mathcal{C}_n$

Let $\Delta = \bigcup_{n=1}^{\infty} \{ I^- : I \in \mathcal{C}_n \}$

then $\nu(\Delta) = 0$. (by cont of $\psi$)

Given $m \geq 1$, let

$$\mathcal{C}_{m,n} = \{ I \in \mathcal{C}_n : \sup_I \varphi - \inf_I \varphi \geq \tfrac{1}{m} \}$$

then

$$\{t \in (a,b] \setminus \Delta : \varphi \text{ is not cont at } t\}$$

$$\subseteq \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \Big( \bigcup_{I \in \mathcal{C}_{m,n}} I \Big).$$

We halt the proof here for now and pick it up next time.

We will need

**Exercise 1.2.26**

If $\psi$ is non-decreasing on $J$

then a bdd $\varphi$ is $RI(\psi)$ on $J$

iff for every $\epsilon > 0$ $\exists\, \delta > 0$

s.t

$$\sum_{\{I \in \mathcal{C} : \sup_I \varphi - \inf_I \varphi \geq \epsilon\}} \Delta_I \psi < \epsilon.$$

whenever $\mathcal{C}$ is finite, non-overlapping exact cover of $J$ w/ $\|\mathcal{C}\| < \delta$.

If $\varphi$ is $RI(\psi)$ on $J$ then given $\epsilon > 0$

$\exists\ \delta > 0$ s.t.

$$\|\mathcal{C}\| < \delta \Rightarrow \left| \sum_{I \in \mathcal{C}} \left(\sup_I \varphi - \inf_I \varphi\right) \Delta_I \psi \right| < \epsilon^2.$$

so, since

$$\epsilon \sum_{\{I \in \mathcal{C} : \sup_I \varphi - \inf_I \varphi \geq \epsilon\}} \Delta_I \psi \leq \sum_{I \in \mathcal{C}} \left(\sup_I \varphi - \inf_I \varphi\right) \Delta_I \psi$$

we have

$$(*) \quad \sum_{\{I \in \mathcal{C} : \operatorname{osc}(\varphi, I) \geq \epsilon\}} \Delta_I \psi < \epsilon \quad \text{when } \|\mathcal{C}\| < \delta.$$

<u>Conversely</u>.

if, for each $\epsilon > 0$ $\exists\ \delta > 0$ s.t. $(*)$ holds

then since

$$\sum_{I \in \mathcal{C}} \operatorname{osc}(\varphi, I) \Delta_I \psi \leq \epsilon \Delta_J \psi + 2\|\varphi\|_u \sum_{\{I : \operatorname{osc}(\varphi, I) \geq \epsilon\}} \Delta_I \psi$$

we have. $\varphi \in RI(\psi, [a,b])$.