Math 137 11-3-08.

<u>Last time</u>

<u>Semi-lattice (of functions).</u>

$f : E \to (-\infty, \infty] \in \mathcal{L}$

$\Rightarrow f^+ \in \mathcal{L}$ and $f^- \in \mathcal{L}$.

<u>$\mathcal{L}$-system</u> $(K \subseteq \mathcal{L})$.

$\mathcal{L}$ a semi-lattice

$K \subseteq \mathcal{L}$ s.t.

a) $\chi_E \in K$

b) $f, g \in K$ and $\{f = \infty\} \cap \{g = \infty\} = \emptyset$
$\Rightarrow$
$f \leq g \Rightarrow g - f \in K$
and
$g - f \in \mathcal{L} \Rightarrow g - f \in K$.

c) $\alpha, \beta \in [0, \infty)$ $f, g \in K$
$\Rightarrow \alpha f + \beta g \in K$.

d) $\{f_n\}_{n=1}^{\infty} \subseteq K$ $f_n \nearrow f \Rightarrow \begin{cases} f \text{ bdd} \Rightarrow f \in K \\ f \in \mathcal{L} \Rightarrow f \in K. \end{cases}$

Lemma: $\mathcal{C}$ a π-system generating a σ-algebra $\mathcal{B}$ over E.

$\mathcal{L}$ a semi-lattice: $f: E \to ~~(-\infty~~, \infty]$

If K is an $\mathcal{L}$-system s.t.

$\chi_\Gamma \in K \quad \forall \Gamma \in \mathcal{C}$

then $\mathcal{L} \cap \{\text{msble on}(E, \mathcal{B})\} \subseteq K$.

Lemma: $(E_1, \mathcal{B}_1)$ $(E_2, \mathcal{B}_2)$ msble spaces

$f: (E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2) \longrightarrow \overline{\mathbb{R}}$ msble.

- For each $x_1 \in E_1$, $f(x_1, \cdot)$ is msble on $(E_2, \mathcal{B}_2)$
- " " $x_2 \in E_2$, $f(\cdot, x_2)$ " " " $(E_1, \mathcal{B}_1)$

If μ_i $i=1,2$ are finite measures on E_i $i=1,2$. (resp.).
then for every msble f, bdd or non-negative.
on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$

$\int_{E_2} f(\cdot, x_2)\, d\mu_2(x_2)$ is msble on E_1

$\int_{E_1} f(x_1, \cdot)\, d\mu_1(x_1)$ is msble on E_2.

Pf: Since every bdd f differs by a constant from a non-neg. function we may assume f is non-neg.

The collection of non-neg fcns on $E_1 \times E_2$ is a semi-lattice $\mathcal{L}$.

Let $K \subseteq \mathcal{L}$ be the collection of functions which meet the requirements of the theorem

if $\Gamma_i \in \mathcal{B}_i$ $\quad i = 1, 2.$ then

$$\chi_{\Gamma_1 \times \Gamma_2}(x_1, \cdot) = \begin{cases} 0 & x_1 \notin \Gamma_1 \\ \chi_{\Gamma_2}(x_2) & x_1 \in \Gamma_1 \end{cases}$$

is a msbl fcn. of $x_2 \in E_2$ for each $x_1 \in E_1$.
And similarly $\chi_{\Gamma_1 \times \Gamma_2}(\cdot, x_2)$ is a msbl function of $x_1 \in E_1$ for each $x_2 \in E_2$.

~~So we only need to check that K is an $\mathcal{L}$-system.~~

This follows from basic properties

also $\int_{E_2} \chi_{\Gamma_1 \times \Gamma_2}(\cdot, x_2)\, d\mu_2(x_2) = \begin{cases} \mu_2(\Gamma_2) & x_1 \in \Gamma_1 \\ 0 & x_1 \notin \Gamma_1 \end{cases}$

is msble on $(E_1, \mathcal{B}_1)$

and similarly for $\int_{E_1} \chi_{\Gamma_1 \times \Gamma_2}(x_1, \cdot)\, d\mu_1(x_1)$.

So we only need to check that K is an $\mathcal{L}$-system.

This follows from basic properties of the integral and the monotone convergence theorem $\blacksquare$

<u>Lemma</u>: Given a pair $(E_1, \mathcal{B}_1, \mu_1)$ $(E_2, \mathcal{B}_2, \mu_2)$ finite measure spaces.

$\exists!$ measure ν on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$ s.t.

$$\nu(\Gamma_1 \times \Gamma_2) = \mu_1(\Gamma_1)\mu_2(\Gamma_2)$$

$$\forall\ \Gamma_1 \in \mathcal{B}_1,\ \Gamma_2 \in \mathcal{B}_2.$$

If $f \geq 0$ is msble on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$ then

$$\int_{E_1 \times E_2} f(x_1, x_2)\, \nu(dx_1 \times x_2)$$

$$= \int_{E_2}\left(\int_{E_1} f(x_1, x_2)\, d\mu_1(x_1)\right) d\mu_2(x_2)$$

$$= \int_{E_1}\left(\int_{E_2} f(x_1, x_2)\, d\mu_2(x_2)\right) d\mu_1(x_1).$$

<u>Pf:</u> The uniqueness of ν follows from Exercise 3.1.8 and requires the finiteness of the measures.

For existence of ν:

let

$$\nu_{1,2}(\Gamma) = \int_{E_2}\left(\int_{E_1} \chi_\Gamma(x_1, x_2)\, d\mu_1(x_1)\right) d\mu_2(x_2).$$

$$\nu_{2,1}(\Gamma) = \int_{E_1}\left(\int_{E_2} \chi_\Gamma(x_1, x_2)\, d\mu_2(x_2)\right) d\mu_1(x_1).$$

for $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$.

Countable additivity for $\nu_{1,2}$ and $\nu_{2,1}$ follow from the monotone convergence theorem and the other properties of a (finite) measure are clear.

So $\nu_{1,2}$, $\nu_{2,1}$ are finite measures on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$.

With $\mathcal{L} = \{f \geq 0\}$ on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$ the set of f for which

$$(*) \quad \int f \, d\nu_{1,2} = \int_{E_1} \left(\int_{E_2} f(x_1, x_2) \, d\mu_1(x_1) \right) d\mu_2(x_2)$$

is a K system containing each $\chi_{\Gamma_1 \times \Gamma_2}$ with $\Gamma_i \in \mathcal{B}_i$ $i = 1, 2$.

and similarly for the set of f with

$$(**) \quad \int f \, d\nu_{2,1} = \int_{E_2} \left(\int_{E_1} f(x_1, x_2) \, d\mu_2(x_2) \right) d\mu_1(x_1).$$

So $(*)$ and $(**)$ hold for all $f \geq 0$ msble in $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$.

Since $\nu_{1,2}(\Gamma_1 \times \Gamma_2) = \mu_1(\Gamma_1)\mu_2(\Gamma_2) = \nu_{2,1}(\Gamma_1 \times \Gamma_2)$

$\forall \; \Gamma_i \in \mathcal{B}_i \quad i = 1, 2.$

we must have

$$\nu_{1,2} = \nu_{2,1} = \nu$$

$\blacksquare$

Def: $(E, \mathcal{B}, \mu)$ is a σ-finite measure space and μ is σ-finite if $E = \bigcup_{n=1}^{\infty} E_n$ s.t.

$$\mu(E_n) < +\infty.$$

Theorem: The conclusions of the previous two lemmas hold on ~~σ-finite~~ products of σ-finite measure spaces.

<u>Pf:</u> Let $\{E_{i,n}\}_{n=1}^{\infty} \subseteq \mathcal{B}_i$ for $i=1,2$.

s.t $\quad E_i = \bigcup_{n=1}^{\infty} E_{i,n}$ and $\mu(E_{i,n}) < \infty$ $\forall n, i$.

W.m.a. $\quad E_{i,m} \cap E_{i,n} = \emptyset \quad m \neq n$ $\quad i=1,2$.

For $n \in \mathbb{Z}^+$ let

$$\mu_{i,n}(\Gamma_i) = \mu_i(\Gamma_i \cap E_{i,n}) \qquad \Gamma_i \in \mathcal{B}_i. \quad i=1,2$$

For $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ let

$\nu_{m,n}$ be the product measure

$\mu_{1,m} \times \mu_{2,n}$ as constructed in

the last lemma.

We have, for any $f \geq 0$, msble on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$

$$\int_{E_2} f(\cdot, x_2)\,d\mu_2(x_2) = \sum_{n=1}^{\infty} \int_{E_{2,n}} f(\cdot, x_2)\,d\mu_{2,n}(x_2)$$

and by our earlier lemma, the L.H.S. is msble on $(E_1, \mathcal{B}_1)$.

Similarly

$$\int_{E_1} f(x_1, \cdot)\,d\mu_1(x_1) \quad \text{is msble on } (E_2, \mathcal{B}_2).$$

Define

$$\nu_0(\Gamma) = \sum_{m,n=1}^{\infty} \nu_{m,n}(\Gamma) \qquad \Gamma \in \mathcal{B}_1 \times \mathcal{B}_2 \ldots$$

The sets $E_{1,k} \times E_{2,j} = A_{k,j}$ are disjoint and $E_1 \times E_2 = \bigcup_{k,j} A_{k,j}$. so it is easy to see that ν_0 is a measure and

$$\nu_0(\Gamma_1 \times \Gamma_2) = \sum_{m,n=1}^{\infty} \nu_{m,n}(\Gamma_1 \times \Gamma_2) = \sum_{m,n=1}^{\infty} \mu_{1,m}(\Gamma_1)\mu_{2,n}(\Gamma_2)$$

$$= \mu_1(\Gamma_1)\mu_2(\Gamma_2)$$

also

$$\int_{E_1 \times E_2} f(x_1, x_2)\, d\nu_0(x_1, x_2)$$

$$= \sum_{k,j} \int_{A_{k,j}} f(x_1, x_2)\, d\nu_0(x_1, x_2)$$

$$= \sum_{k,j} \int_{A_{k,j}} f(x_1, x_2)\, d\nu_{k,j}(x_1, x_2).$$

and it follows from the previous lemma that the product integral is equal to each of the iterated integrals.

If ν is any other measure satisfying the requirements of the theorem then

$(\nu(P_1 \times P_2) = \mu_1(P_1)\mu_2(P_2))$

by uniqueness in the finite case

$\nu = \nu_{m,n}$ when restricted to $A_{m,n}$. $\forall (m,n)$

$\therefore \quad \nu = \nu_0$ on $\mathcal{B}_1 \times \mathcal{B}_2$

$\boxtimes$

Theorem: (Fubini's Theorem).

$(E_1, \mathcal{B}_1, \mu_1)$
$(E_2, \mathcal{B}_2, \mu_2)$ σ-finite msre spaces.

f msble on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$

is $\mu_1 \times \mu_2$ integrable

$$\Longleftrightarrow \int_{E_1} \left[\int_{E_2} |f(x_1, x_2)|\, d\mu_2(x_2)\right] d\mu_1(x_1) < \infty$$

$$\Longleftrightarrow \int_{E_2} \left[\int_{E_1} |f(x_1, x_2)|\, d\mu_1(x_1)\right] d\mu_2(x_2) < \infty.$$

(This is an immediate consequence of Tonelli's thm).

Now put

$$\Lambda_1 = \left\{ x_1 \in E_1 : \int_{E_2} |f(x_1, x_2)|\, d\mu_2(x_2) < \infty \right\}$$

$$\Lambda_2 = \left\{ x_2 \in E_2 : \int_{E_1} |f(x_1, x_2)|\, d\mu_1(x_1) < \infty \right\}.$$

then (by Tonelli again) $\Lambda_i \in \mathcal{B}_i$ $i = 1, 2$.

So

$$f_1(x_1) \equiv \begin{cases} \int_{E_2} f(x_1, x_2)\, d\mu_2(x_2) & x_1 \in \Lambda_1 \\ 0 & x_1 \in \Lambda_1^c \end{cases}$$

and

$$f_2(x_2) \equiv \begin{cases} \int_{E_1} f(x_1, x_2)\, d\mu_1(x_1) & x_2 \in \Lambda_2 \\ 0 & x_2 \in \Lambda_2^c \end{cases}$$

are $\mathbb{R}$ valued msble functions on E_i $i=1,2$ resp. (by our previous lemma applied to f^+, f^-.

If f is $\mu_1 \times \mu_2$ integrable then (by the 1st assertion) $\mu_i(\Lambda_i^c)=0$ and $f_i \in L^1(\mu_i)$. and we have

$$(*) \qquad \int_{E_i} f_i(x_i)\, d\mu_i(x_i) = \int_{E_1 \times E_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2)$$

$i=1,2.$

Pf: The only unproved assertion is (*).

We have $\mu_1 \times \mu_2(\Omega_1^c \times E_2) = \mu_1(A_1^c)\mu_2(E_2) = 0$

so

$$\int_{E_1 \times E_2} f(x_1, x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2)$$

$$= \int_{\Omega_1 \times E_2} f^+(x_1, x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2)$$

$$- \int_{\Omega_1 \times E_2} f^-(x_1, x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2)$$

$$= \int_{\Omega_1} \left(\int_{E_2} f^+(x_1, x_2) \, d\mu_2(x_2) \right) d\mu_1(x_1)$$

$$- \int_{\Omega_1} \left(\int_{E_2} f^-(x_1, x_2) \, d\mu_2(x_2) \right) d\mu_1(x_1)$$

$$= \int_{\Omega_1} f_1(x_1) \, d\mu_1(x_1).$$

and similarly

$$\int_{E_1 \times E_2} f(x_1, x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2) = \int_{A_2} f_2(x_2) \, d\mu_2(x_2).$$