Math 137 10-29-08

<u>Theorem</u>: Kolmogorov's Strong Law

$\{X_n\}_{n=1}^{\infty}$ i.i.d on $(\Omega, \mathcal{F}, P)$.

$X_1 \in L^1(P)$ $\quad E(X_1) = m$

$\Rightarrow \bar{S}_n \to m$ P a.s. and in $L^1(P)$.
as $n \to \infty$.

Conversely, if $\bar{S}_n$ converges (in $\mathbb{R}$)

on a set A with $P(A) > 0$

then $X_1 \in L^1(P)$

<u>Pf</u>: Suppose $X_1 \in L^1(P)$ and assume as we may that $E(X_1) = 0$.

Let $f_n(t) = t\,\chi_{[-n,n]}(t)$ and note that

$$\sum_{n=1}^{\infty} \mathrm{Prob}\{X_n \neq f_n(X_n)\} = \sum_{n=1}^{\infty} P\{|X_n| > n\}$$

$$\leq \sum_{n=1}^{\infty} \int_{n-1}^{n} P\{|X_n| > t\}\,dt$$

$$= \sum_{n=1}^{\infty} \int_{n-1}^{n} P\{|X_1| > t\}\,dt$$

$$= E(|X_1|)$$

By Borell - Cantelli part 1, with Prob = 1 only finitely many of the events $\{X_n \neq f_n(X_n)\}$ occur.

i.e $\text{Prob}\{\exists n \in \mathbb{Z}^+ : \forall N \geq n \quad X_N = f_N(X_N)\}$

$$= 1.$$

$$\therefore \quad \frac{1}{n}\sum_{l=1}^{n} X_l \to 0 \qquad \text{a.e.}$$

$$\Leftrightarrow \quad \frac{1}{n}\sum_{l=1}^{n} f_l(X_l) \to 0. \qquad \text{a.e.}$$

We will now show that

$$\frac{1}{n}\sum_{l=1}^{n} f_l(X_l) \to 0 \qquad \text{a.e.}$$

From <u>last time</u>, we have

for any $\{b_n\}_{n=1}^{\infty}$ with $b_n \nearrow +\infty$

$\mathcal{X}_n$, ind. w/ $E(\mathcal{X}_n^2) < +\infty$.

s.t $\quad \sum_{n=1}^{\infty} \frac{\text{Var}(\mathcal{X}_n)}{b_n^2} < \infty$

then $\quad \frac{1}{b_n}\sum_{l=1}^{n}(\mathcal{X}_l - E^P[\mathcal{X}_l]) \to 0$ P a.s.

and we have, for $f_\ell(X_\ell)$

$$\frac{1}{n} \sum_{\ell=1}^{n} E^P(f_\ell(X_\ell))$$

From Math 151:

$$\left(\begin{aligned} &E^P(f_\ell(X_\ell)) \\ &= \int_{-\infty}^{\infty} f_\ell(x) \underbrace{h_{X_\ell}(x)\,dx}_{\substack{\| \\ h_{X_1}(x)\,dx \\ \| \\ \text{p.d.f of } X_1}} \end{aligned} \right).$$

$$= \frac{1}{n} \sum_{\ell=1}^{n} E^P(X_\ell, |X_\ell| \le \ell)$$

$$= \frac{1}{n} \sum_{\ell=1}^{n} E^P(X_1, |X_1| \le \ell).$$

$$\longrightarrow \lim_{n\to\infty} E^P(X_1, |X_1| \le n)$$

Again, we will see this rigorously in Chapt 5 of [CI]

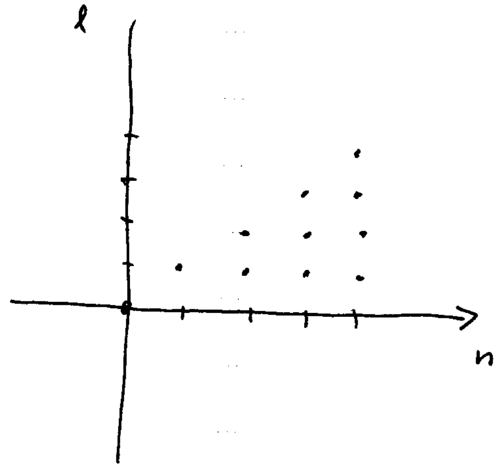
$$= E(X_1) = 0.$$

So by our result from last time it suffices to show that

$$\sum_{n=1}^{\infty} \frac{E^P(f_n(X_n)^2)}{n^2} < +\infty.$$

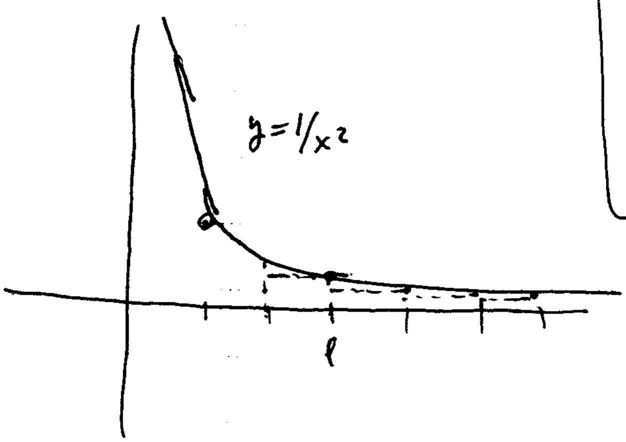
but,

$$\sum_{n=1}^{\infty} \frac{E^P(f_n(X_n)^2)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\ell=1}^{n} E^P[X_1^2, \ell-1<|X_1|\le \ell]$$



$$= \sum_{\ell=1}^{\infty} \left(\sum_{n=\ell}^{\infty} \frac{1}{n^2}\right) E^P[X_1^2, \ell-1<|X_1|\le \ell]$$

$$\boxed{\sum_{n=\ell}^{\infty} \frac{1}{n^2} \le \int_{\ell-1}^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{\ell-1}^{\infty} = \frac{1}{\ell-1} \le \frac{C}{\ell}.}$$



$$\le C \sum_{\ell=1}^{\infty} \frac{1}{\ell} E^P[X_1^2, \ell-1<|X_1|\le \ell]$$

$$\le C \sum_{\ell=1}^{\infty} E^P[|X_1|, \ell-1<|X_1|\le \ell]$$

$$\le C\, E^P[|X_1|] < +\infty.$$

$\therefore \quad \bar{S}_n \to m \quad P$ a.s.

and we have the $L^1(P)$ convergence from the weak law.

For the converse, recall that by our lemma from last time on convergence and tail events:

if $\bar{S}_n$ converges on a set of positive measure then it converges almost surely to a constant value m.

$\therefore$

$$\overline{\lim_{n\to\infty}} \frac{|X_n|}{n} = \overline{\lim_{n\to\infty}} \left| \bar{S}_n - \bar{S}_{n-1} \right| = 0 \quad \text{a.s.}$$

and

$\therefore$ with $A_n = \{ |X_n| > n \}$

we have $P(\overline{\lim_{n\to\infty}} A_n) = 0$.

(with prob = 1 only finitely many A_n occurs).

The second part of Borel Cantelli says that if A_n are iid and $\sum P(A_n) = +\infty$ then $P(\overline{\lim_{n\to\infty}} A_n) = \underline{1}$.

Since our A_n are independent we must have.

$$\sum_{n=1}^{\infty} P\{ |X_n| > n \} < +\infty .$$

$$\therefore\ E^P[|X_1|] = \int_0^\infty P(|X_1| > t)\,dt$$

$$\leq 1 + \sum_{n=1}^{\infty} P(|X_n| > n) < \infty.$$

$\blacksquare$

$(E_1, \mathcal{B}_1)$, $(E_2, \mathcal{B}_2)$ msble spaces.

$\mathcal{B}_1 \times \mathcal{B}_2 \equiv \sigma(E_1 \times E_2 ; \{\Gamma_1 \times \Gamma_2 : \Gamma_1 \in \mathcal{B}_1, \Gamma_2 \in \mathcal{B}_2\})$

<u>Def</u>: The <u>product</u> of $\underline{(E_1, \mathcal{B}_1)}$, $\underline{(E_2, \mathcal{B}_2)}$

is $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$.

Given μ_i, a measure on $(E_i, \mathcal{B}_i)$ $i=1,2$

we want to ~~construct a show the exi~~ construct a measure ν on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$

s.t $\nu(\Gamma_1 \times \Gamma_2) = \mu_1(\Gamma_1)\mu_2(\Gamma_2)$

$\forall$ $\Gamma_1 \in \mathcal{B}_1$, $\Gamma_2 \in \mathcal{B}_2$.

Such a measure, if ~~it~~ one exists, will be unique since the collection of $\Gamma_1 \times \Gamma_2$ is a generating π-system for $\mathcal{B}_1 \times \mathcal{B}_2$. $\big((\Gamma_1 \times \Gamma_2) \cap (\Gamma_3 \times \Gamma_4) = (\Gamma_1 \cap \Gamma_3) \times (\Gamma_2 \cap \Gamma_4)\big)$

the same is not true ~~for~~ when $\cap$ is replaced by $\cup$.

To extend the definition of our (potentially good) ν from sets $\Gamma_1 \times \Gamma_2$ $\Gamma_1 \in \mathcal{B}_1$, $\Gamma_2 \in \mathcal{B}_2$ to all of $\mathcal{B}_1 \times \mathcal{B}_2$, we need a type of π-λ lemma for functions.

<u>Def</u>: $\mathcal{L}$, a collection of functions on a space E.

$f: E \to (\infty, \infty]$ is a

<u>semi-lattice</u> if

$f \in \mathcal{L} \Rightarrow f^+ \in \mathcal{L}$ and $f^- \in \mathcal{L}$.

<u>with $\mathcal{L}$ as above</u>

<u>Def</u>: $K \subset \mathcal{L}$ is an $\mathcal{L}$ system if

a) $\chi_E \in K$

b) if $f, g \in K$, $\{f = \infty\} \cap \{g = \infty\} = \emptyset$ then

$f \le g \Rightarrow g - f \in K$

and $g - f \in \mathcal{L} \Rightarrow g - f \in K$.

c) $\alpha, \beta \in [0, \infty)$, $f, g \in K \Rightarrow \alpha f + \beta g \in K$

d) if $\{f_n\}_{n=1}^{\infty} \subseteq K$ $\quad f_n \nearrow f$ $\quad$ then

$$f \text{ bdd} \Rightarrow f \in K.$$

and

$$f \in \mathcal{L} \Rightarrow f \in K.$$

<u>Lemma</u>: $\mathcal{C}$ a π-system which generates the σ-algebra $\mathcal{B}$ over E.

$\mathcal{L}$ a semi-lattice of functions $f : E \to [-\infty, \infty]$.

If K is an $\mathcal{L}$ system and $\chi_\Gamma \in K$ for each $\Gamma \in \mathcal{C}$, then K contains every $f \in \mathcal{L}$ which is measurable on $(E, \mathcal{B})$.

<u>Pf:</u>

Pf: let $\mathcal{H} = \{\Gamma \subseteq E : \chi_\Gamma \in \mathcal{K}\}$

- $\mathcal{C} \subset \mathcal{H}$.
- if $\Gamma_1, \Gamma_2 \in \mathcal{H}$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$ then

  by c) $\chi_{\Gamma_1} + \chi_{\Gamma_2} = \chi_{\Gamma_1 \cup \Gamma_2} \in K$.

  so $\Gamma_1 \cup \Gamma_2 \in \mathcal{H}$.

- if $\Gamma_1, \Gamma_2 \in \mathcal{H}$ w/ $\Gamma_1 \subseteq \Gamma_2$ then by b)

  $$\chi_{\Gamma_2 \setminus \Gamma_1} = \chi_{\Gamma_2} - \chi_{\Gamma_1} \in K$$

  so $\Gamma_2 \setminus \Gamma_1 \in \mathcal{H}$.

- if $\Gamma_n \nearrow \Gamma$. then by d) $\chi_\Gamma \in K$.

  so $\Gamma \in \mathcal{H}$.

$\therefore$ $\chi_\Gamma \in K \quad \forall\, \Gamma \in \mathcal{B}$.

and by c) $\mathcal{K}$ contains every non-negative msble, simple function on $(E, \mathcal{B})$.

If f is non-negative and msble then $f \in K$ by d) since any such f is a monotone limit of ~~[illegible]~~ non-neg msble simple functions.

if $f \in \mathcal{L}$ is mble then $f = f^+ - f^-$.

$f^+, f^- \in \mathcal{L}$ (by def. of $\mathcal{L}$).

so by b) $f \in K$.

<u>Lemma:</u> $(E, \mathcal{B}_1)$, $(E_2, \mathcal{B}_2)$ mble spaces.

$f: (E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2) \longrightarrow \overline{\mathbb{R}}$ is mble.

- for each $x_1 \in E_1$ and each $x_2 \in E_2$

  $f(x_1, \cdot)$, $f(\cdot, x_2)$ are mble

  on $E_2\, \mathcal{B}_2$, $E_1\, \mathcal{B}_1$ (resp.).

- if μ_i is a finite measure on $(E_i, \mathcal{B}_i)$ $i=1,2$

  and f is either bdd or non-neg and mble on $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$

  then

  $\int_{E_2} f(\cdot, x_2)\, d\mu_2(x_2)$ is mble on $(E_1, \mathcal{B}_1)$.

  and $\int_{E_1} f(x_1, \cdot)\, d\mu_1(x_1)$ is mble on $(E_2, \mathcal{B}_2)$

So $\chi_{P_1 \times P_2} \in K \quad \forall\ P_1 \in \mathcal{B}_1,\ P_2 \in \mathcal{B}_2$.

and (by the previous lemma with $\mathcal{C} = \{P_1 \times P_2 ;\ P_1 \in \mathcal{B}_1, P_2 \in \mathcal{B}_2\}$) we just need to check that K is an $\mathcal{L}$-system.

$\chi_{E_1 \times E_2} \in K$ has been checked.

and the other 3 properties of an $\mathcal{L}$ system are simple consequences of previous results on msble functions.