Math 137 10-27-08

Strong Law of Large Numbers

$\{X_n\}_{n=1}^{\infty}$ are ind. r.v's on $(\Omega, \mathcal{F}, P)$

Lemma: For any sequences

$$\{a_n : n \in \mathbb{Z}^+\}, \quad \{b_n : n \in \mathbb{Z}^+\}$$

s.t $\{b_n\}$ has a limit in $[0, \infty]$,

the set

$$\left\{x : \lim_{n \to \infty} \frac{S_n - a_n}{b_n} \text{ exists in } \mathbb{R}\right\}$$

has P measure 0 or 1.

If $b_n \to \infty$ as $n \to \infty$

then

$$\overline{\lim_{n \to \infty}} \frac{S_n - a_n}{b_n} \quad \text{and} \quad \underline{\lim_{n \to \infty}} \frac{S_n - a_n}{b_n}$$

are constant P a.e.

Pf: $\left\{\lim \frac{S_n - a_n}{b_n} \text{ exists in } \mathbb{R}\right\} = \left\{\lim_{m \to \infty} \sup_{n \geq m} \left| \frac{S_{n+m} - a_{n+m}}{b_{n+m}} - \frac{S_m - a_m}{b_m} \right| = 0\right\}$

is a tail event int the tail generated by $\{\sigma(X_n)\}_{n=1}^{\infty}$

If $b_n \to \infty$ then

$$\overline{\lim_{n\to\infty}} \left[\frac{S_{n+m} - S_m - a_{n+m}}{b_{m+n}} \right] = \overline{\lim} \left(\frac{S_n - a_n}{b_n} \right)$$

and $$\underline{\lim_{n\to\infty}} \left[\frac{S_{n+m} - S_m - a_{n+m}}{b_{m+n}} \right] = \underline{\lim} \frac{S_n - a_n}{b_n}$$

So for any c and $\epsilon > 0$ and $m_0 \in \mathbb{Z}^+$

$$\left\{ c < \overline{\lim} \left(\frac{S_n - a_n}{b_n} \right) < c + \epsilon \right\}$$

$$= \bigcap_{m \geq m_0} \left\{ c < \overline{\lim} \left(\frac{S_{n+m} - S_m - a_{n+m}}{b_{m+n}} \right) < c + \epsilon \right\}$$

and $\therefore$ is a tail event.

Similarly for $\underline{\lim}$.

<u>Theorem</u>: If the X_n's are independent, square P-integrable and

$$\sum_{n=1}^{\infty} Var(X_n) < \infty$$

then $\sum_{n=1}^{\infty} (X_n - E^P(X_n))$ converges P-almost surely.

Note that

$$\sup_{n \geq N} P\left(\left| \sum_{\ell=N}^{n} X_\ell - E^P[X_\ell] \right| \geq \epsilon \right)$$

$$\leq \frac{1}{\epsilon^2} \sum_{\ell=N}^{\infty} Var(X_\ell)$$

since

$P(|\cdot| \geq \epsilon) \leq \frac{1}{\epsilon^2} \int |\cdot|^2 \, dP$ and cross terms vanish by independence.

Our Cauchy-criteria for convergence in measure shows that

$$\sum_{n=1}^{\infty} (X_n - E^P(X_n))$$ converges

in P-probability.

If we can bring the "$\sup_{n \geq N}$" inside the $P(\quad)$ then we will have our Cauchy criteria for a.e. convergence.

Theorem: (Kolmogorov's inequality)

If X_n's are independent and square P-integrable then

$$P\left(\sup_{n \geq 1} \left| \sum_{\ell=1}^{n} (X_\ell - E^P(X_\ell)) \right| \geq \epsilon \right) \leq \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} Var(X_n)$$

Pf: W.L.O.G. W.M.A. $E(X_n) = 0 \quad \forall n.$

If $1 \leq n < N$ then

$$S_N^2 - S_n^2 = (S_N - S_n)^2 + 2(S_N - S_n)S_n$$

$$\geq 2(S_N - S_n)S_n.$$

Now $E(S_N - S_n) = 0$ and $S_N - S_n$ is independent of $\sigma(X_1, \dots, X_n)$, so if $A \in \sigma(X_1, \dots, X_n)$ then

$$E((S_N^2 - S_n^2)\chi_{A_n}) \geq E(2(S_N - S_n)S_n \chi_A)$$

$$= 0.$$

and $\therefore$

$$E(S_N^2, A_n) \geq E(S_n^2, A_n)$$

for any $A_n \in \sigma(X_1, \dots, X_n)$.

Let $A_1 = \{|S_1| > \epsilon\}$

$\vdots$

$$A_{n+1} = \{|S_{n+1}| > \epsilon \text{ and } \max_{1 \leq l \leq n} |S_l| \leq \epsilon\}$$

$$n \in \mathbb{Z}^+$$

then $A_m \cap A_n = \emptyset \qquad m \neq n$

and

$$B_N \equiv \{\max_{1 \leq n \leq N} |S_n| > \epsilon\} = \bigcup_{n=1}^{N} A_n$$

so

$$E^P[S_N^2, B_N] = \sum_{n=1}^{N} E^P[S_N^2, A_n]$$

$$\geq \sum_{n=1}^{N} E^P[S_n^2, A_n]$$

$$\geq \epsilon^2 \sum_{n=1}^{N} P(A_n) = \epsilon^2 P(B_N)$$

in particular,

$$\epsilon^2 P(\sup_{n \geq 1} |S_n| > \epsilon) = \lim_{N \to \infty} \epsilon^2 P(B_N) \leq \lim_{N \to \infty} E^P[S_N^2] \leq \sum_{n=1}^{\infty} E^P(X_n^2).$$

As $\epsilon \nearrow \epsilon_0$

$$\left(\sup_{n\geq 1} |S_n| > \epsilon\right) \searrow \left(\sup_{n\geq 1} |S_n| \geq \epsilon_0\right).$$

so by monotonicity of measures (P)

$$\epsilon^2 P\left(\sup_{n\geq 1} |S_n| \geq \epsilon\right) \leq \sum_{n=1}^{\infty} E^P(X_n^2) \quad \blacksquare$$

To prove the 1st theorem, w.m.a.

$E^P(X_n) = 0 \quad \forall n$ and apply

Kolmogorov's inequality to

$\{X_{N+n} : n \in \mathbb{Z}^+\}$ to get

$$P\left(\sup_{n>N} |S_n - S_N| \geq \epsilon\right) \leq \frac{1}{\epsilon^2} \sum_{n=N+1}^{\infty} E^P(X_n^2)$$

Since $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) = \sum_{n=1}^{\infty} E(X_n^2) < +\infty$

we have $P\left(\sup_{n>N} |S_n - S_N| \geq \epsilon\right) \xrightarrow[N\to\infty]{} 0$

which is our Cauchy Criteria for a.e. convergence.

<u>Lemma:</u> Let $\{b_n\}_{n=1}^{\infty}$ be non-decreasing with $\lim_{n\to\infty} b_n = +\infty$ and put $\beta_n = b_n - b_{n-1}$, $b_0 \equiv 0$.

part 1: If $\{s_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ has $s_n \to s \in \mathbb{R}$, $n \to \infty$.

then

$$\frac{1}{b_n} \sum_{\ell=1}^{n} \beta_\ell s_\ell \longrightarrow s.$$

part 2: This implies that if $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$

then

$$\sum_{n=1}^{\infty} \left(\frac{x_n}{b_n}\right) \text{ converges in } \mathbb{R} \Rightarrow \frac{1}{b_n} \sum_{\ell=1}^{n} x_\ell \to 0 \quad n \to \infty.$$

<u>Pf:</u> (part 1) w.m.a. $s=0$.

Given $\epsilon > 0$, choose $N \in \mathbb{Z}^+$ s.t.

$\ell \geq N \Rightarrow |s_\ell| < \epsilon$.

Let $M = \sup_{n \geq 1} |s_n|$. Then

$$\frac{1}{b_n} \sum_{\ell=1}^{n} \beta_\ell s_\ell = \frac{1}{b_n}\left(\sum_{\ell=1}^{N} \beta_\ell s_\ell + \sum_{\ell=N+1}^{n} \beta_\ell s_\ell\right) \leq \frac{M(b_N - b_0)}{b_n} + \epsilon \frac{b_n - b_N}{b_n}$$

which shows that

$$\left|\frac{1}{b_n}\sum_{\ell=1}^{n}\beta_\ell s_\ell\right| < 2\epsilon \quad \text{if } n \text{ is suff. large.}$$

For the second part, we use the trick of summation by parts

recall:

$$\{a_j\}_{j=1}^{\infty} \quad \{b_k\}_{k=1}^{\infty}$$

$$\sum_{k=m}^{N} a_k b_k = a_N B_N - a_m B_{m-1} - \sum_{k=m}^{N-1} B_k (a_{k+1} - a_k)$$

where $B_0 = 0$

$$B_k = \sum_{j=1}^{k} b_j.$$

We have:

$$\sum_{\ell=1}^{n} x_\ell = \sum_{\ell=1}^{n} b_\ell \left(\frac{x_\ell}{b_\ell}\right) = b_n \sum_{\ell=1}^{n}\frac{x_\ell}{b_\ell} - b_1 \cdot 0 - \sum_{\ell=1}^{n-1} s_\ell \beta_{\ell+1}$$

$$= b_n \cdot s_n - \sum_{j=1}^{n} s_{j-1}\cdot \beta_j \qquad (s_0 \equiv 0).$$

So

$$\frac{1}{b_n}\sum_{\ell=1}^{n} x_\ell = s_n - \frac{1}{b_n}\sum_{j=1}^{n} s_{j-1}\beta_j \longrightarrow 0$$

(by part 1.).

Corollary (to the theorem).

Suppose $\{b_n\} \nearrow +\infty \quad n \to \infty$

and is positive and finite

Suppose $\{X_n\}_{n=1}^{\infty}$ are independent

$$E^P(X_n^2) < +\infty$$

If $$\sum_{n=1}^{\infty} \frac{\operatorname{var}(X_n)}{b_n^2} < \infty$$

$$\left(= \sum_{n=1}^{\infty} \operatorname{Var}\left(\frac{X_n}{b_n}\right) \right).$$

then

$$\frac{1}{b_n} \sum (X_\ell - E^P[X_\ell]) \to 0 \quad P \text{ a.s.}$$

Consequently,

if $\{X_n\}$ are i.i.d and $E(X_n^2) < \infty$

then

$$\overline{S}_n \to m = E(X_1) \quad P\text{-a.s.}$$

Thm: Kolmogorov's Strong Law

X_n ind. id. dist. $(n \geq 0)$
(i.i.d).

$X_1 \in L^1(P)$ $\qquad E(X_1) = m$

$\Rightarrow \bar{S}_n \to m \quad P \text{ a.s}, \quad n \to \infty$
and in $L^1(P)$.

Conversely,
If $\bar{S}_n$ converges on a set of positive measure then $X_1 \in L^1(P)$.

For the proof, we require the following fact which is a special case of the main result in [CI] 5.1.

$$* \quad E^P[\,|X_1|\,] = \int_0^\infty P(\,|X_1| > t\,)\,dt.$$

If we know (*) for bdd X_1 then we would have

$$E^P\left[|f_m \circ X_1|\right] = \int_0^\infty P(|f_m \circ X_1| > t)\, dt.$$

where $f_m(t) = t \cdot \chi_{[-m,m]}(t)$

and the general form of (*) would follow by monotone convergence.

So we may assume X_1 has $|X_1| \leq M$.

Then

$$\int_0^\infty P(|X_1| > t)\, dt = \int_0^M P(|X_1| > t)\, dt.$$

The function $P(|X_1| > t) = \omega(t)$

is decreasing, bdd, continuous on the ~~left~~ right. $\therefore$ Riemann integrable and $\therefore$ $\displaystyle\int_0^m P(|X_1|>t)\,dt$

$$\approx \sum_{j=0}^{n-1} P(|X_1|>t_j)(t_{j+1}-t_j)$$

with $t_0=0$, $t_j = j\cdot\frac{m}{n}$

$t_n = m.$

$$= \frac{m}{n}\Big[n\cdot P(|X_1|>t_{n-1}) + n-1\, P(t_{n-2}<|X_1|\le t_{n-1})$$

$$+ \dots + (n-j)\, P(t_{n-j-1}<|X_1|\le t_{n-j})$$

$$+ \dots + 1\cdot P(\underset{\overset{\parallel}{0}}{t_0}<|X_j|\le t_1)\Big]$$

$$= t_n\cdot P(|X_1|>t_{n-1}) + t_{n-1}\, P(t_{n-2}<|X_1|\le t_{n-1})$$

$$+ \dots + t_{(n-j)}\, P(t_{n-j-1}<|X_1|\le t_{n-j})$$

$$+ \dots + t_1\, P(t_0<|X_1|\le t_1]$$

$$\longrightarrow E^P(|X_1|) \quad \text{as } n\to\infty.$$

Pf: (Theorem).

Suppose X_1 is P integrable and $E(X_1)=0$.

Let $Y_n = X_n \, \chi_{[-n,n]}(X_n)$.

Then

$$\sum_{n=1}^{\infty} P(Y_n \neq X_n) = \sum_{n=1}^{\infty} P(|X_n| > n)$$

$$\leq \sum_{n=1}^{\infty} \int_{n-1}^{n} P(|X_1| > t)\,dt$$

$$= \int_0^{\infty} P(|X_1| > t)\,dt = E^P[|X_1|] < +\infty.$$

By 1st part of Borel Cantelli

$\text{Prob}\{X_n \neq Y_n \text{ for infinitely many } n\} = 0.$

i.e. $\text{Prob}\{X_n = Y_n \text{ for all but finitely many } n\} = 1.$

So if $\overline{T_n} = \frac{1}{n}\sum_{\ell=1}^{n} Y_\ell$ then for

P a.e. ω $\overline{T_n}(\omega) \to 0 \iff \overline{S_n}(\omega) \to 0.$

We claim that $\bar{T}_n \to 0$ P a.s.:

$$\frac{1}{n}\sum_{\ell=1}^{n} E^P[Y_\ell] = E^P[X_1, |X_1| \le n]$$

so $\lim_{n\to\infty} \frac{1}{n}\sum_{\ell=1}^{n} E^P[Y_\ell] = 0$.

and by our previous corollary we just need to check that

$$\sum_{n=1}^{\infty} \frac{E^P(Y_n^2)}{n^2} < +\infty .$$

but

$$\sum_{n=1}^{\infty} \frac{E^P(Y_n^2)}{n^2} = \sum_{n=1}^{\infty} \sum_{\ell=1}^{n} E^P[X_1^2, \ell-1<|X_1|\le \ell]$$

$$= \sum_{\ell=1}^{\infty} E^P[X_1^2, \ell-1<|X_1|\le\ell] \sum_{n=\ell}^{\infty}\frac{1}{n^2}$$

$$\le \cancel{\sum} \sum_{\ell=1}^{\infty} \frac{C}{\ell} E^P[X_1^2, \ell-1<|X_1|\le\ell]$$

$$\le C \sum_{\ell=1}^{\infty} E^P[|X_1|, \ell-1<|X_1|\le\ell]$$

$$\le C\, E^P[|X_1|] < \infty .$$

So $\overline{S}_n \to 0$ P-a.s. and the $L^1(P)$ convergence comes from the weak Law we have already proved.

For the converse, note that our first lemma $\Rightarrow$ $\overline{S}_n$ converges. a.s. if it converges on a set of positive measure.

$$\therefore \overline{\lim_{n\to\infty}} \frac{|X_n|}{n} = \lim_{n\to\infty} |\overline{S}_n - \overline{S}_{n-1}| = 0$$

P-a.e.

So if $A_n \equiv \{|X_n| > n\}$

then $P(\overline{\lim_{n\to\infty}} A_n) = 0$.

Since then A_n's are mutually independent, the 2nd part of Borel Cantelli shows that

$$\sum_{n=1}^{\infty} P(A_n) < \infty.$$

$\therefore$

$$E^P[|X_1|] = \int_0^\infty P(|X_1| > t)\,dt$$

$$\leq 1 + \sum_{n=1}^\infty P(|X_1| > n) < \infty .$$