Math 137 10-22.

Recall: Convergence in measure

$$f_n \xrightarrow{\mu} f$$

iff for all $\epsilon > 0$

$$\mu\{|f_n - f| > \epsilon\} \to 0 \quad \text{as } n \to \infty.$$

(Theorem) iff

$$\lim_{m\to\infty} \sup_{n \geq m} \mu(|f_n - f_m| \geq \epsilon) = 0 \quad \forall \epsilon > 0.$$

and $\exists\ n_i$ s.t.

$$f_{n_i} \to f \quad \text{a.e.} \quad i \to \infty.$$

We have:

$$\|f_n - f\|_{L^1(\mu)} \to 0 \implies f_n \xrightarrow{d\mu} f$$

$$\implies \lim_{i\to\infty} \mu(\sup_{j \geq i} |f_{n_j} - f| \geq \epsilon) = 0 \quad \epsilon > 0$$

for some subsequence $\{f_{n_i}\}$.

$$\implies f_{n_i} \to f \quad \text{a.e.}$$

also,

$\mu(E) < \infty$ and $f_n \to f$ a.e. $\implies f_n \to f$ in measure.

Our ~~3~~ main convergence theorems all hold with a.e. convergence replaced by convergence in measure.

(recall that if $\mu(E) = +\infty$ a.e convergence does not imply convergence in measure).

Theorem: f, $\{f_n\}_{n=1}^{\infty}$ msble R-ld. on $(E, \mathcal{B}, \mu)$

$f_n \to f$ in measure.

then

Fatou: If $f_n \geq 0$ (a.e) $\forall n$

then $f \geq 0$ a.e.

and $\int f \, d\mu \leq \underline{\lim} \int f_n \, d\mu$

<u>Dominated</u>: If $\exists\ g \in L^1(\mu)$ on $(E, \mathcal{B}, \mu)$

s.t. $|f_n| \le g \qquad n \ge 1$,

then f is integrable

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mu)} = 0$$

$$\therefore \int f_n\, d\mu \to \int f\, d\mu \quad \text{as } n \to \infty.$$

<u>Lieb</u>: If $\sup_{n\ge 1} \|f_n\|_{L^1(\mu)} < \infty$ then

f is integrable and

$$\lim_{n\to\infty} \Big| \|f_n\|_{L^1(\mu)} - \|f\|_{L^1(\mu)} - \|f_n - f\|_{L^1(\mu)} \Big|$$

$$= \lim_{n\to\infty} \Big\| |f_n| - |f| - |f_n - f| \Big\|_{L^1(\mu)} = 0.$$

$$\therefore \quad \|f_n - f\|_{L^1(\mu)} \to 0 \quad \text{iff} \quad \|f_n\|_{L^1(\mu)} \to \|f\|_{L^1(\mu)}.$$

<u>Pf:</u> (Fatou).

Suppose $f_n \ge 0$ a.e.

and $f_n \xrightarrow{\mu} f$

Choose f_{n_m} s.t.

$$\lim_{m\to\infty} \int f_{n_m}\, d\mu \to \underline{\lim}_{n\to\infty} \int f_n\, d\mu$$

then choose

$$f_{n_{m_i}} \quad \text{s.t} \quad f_{n_{m_i}} \to f \quad \underset{i\to\infty}{\text{a.e.}}$$

Since $f_{n_{m_i}} \geq 0$ a.e. we have $f \geq 0$ a.e.

Ignoring a countable union of sets of measure zero w.m.a.

$$f_{n_{m_i}} \geq 0, \quad f \geq 0 \quad \text{and}$$

$$f_{n_{m_i}} \to f \quad \text{everywhere.}$$

So. (by Fatou's lemma).

$$\int f\, d\mu \leq \underline{\lim}_{i} \int f_{n_{m_i}}\, d\mu = \lim_{m\to\infty} \int f_{n_m}\, d\mu$$

$$= \underline{\lim}_{n\to\infty} \int f_n\, d\mu$$

$\boxtimes$

<u>Corollary</u> $L^1(\mu)$ is complete.

i.e. Let $\{f_n\} \subseteq L^1(\mu)$

If $\lim_{m\to\infty} \sup_{n\ge m} \|f_n - f_m\|_{L^1(\mu)} = 0$.

then $\exists\ f \in L^1(\mu)$ s.t.

$\|f_n - f\|_{L^1(\mu)} \to 0$.

<u>Pf:</u> $\mu\{|f_n - f_m| > \epsilon\} \le \frac{1}{\epsilon}\int |f_n - f_m|\, d\mu$.

so $\lim_{m\to\infty} \sup_{n\ge m} \mu\{|f_n - f_m| > \epsilon\} = 0 \quad \forall \epsilon > 0$.

$\therefore \quad f_n \xrightarrow{\mu} f$ for some f

Now by the version of Fatou above.

$$\|f - f_m\|_{L^1(\mu)} \le \varliminf_{n\to\infty} \|f_n - f_m\|_{L^1(\mu)}.$$

$$\le \sup_{n\ge m} \|f_n - f_m\|_{L^1(\mu)} \to 0.$$

$m \to \infty$.

Since
$\|f\|_1 \le \|f - f_n\|_1 + \|f_n\|_1$ and $\|f_n\|$ is unif bdd. $\|f\|$ is integrable

Thm: $(E, \mathcal{B}, \mu)$

$\mu(E) < +\infty$.

$\mathcal{C} \subset P(E)$ is a π-system generating $\mathcal{B}$.

$$S = \left\{ \sum_{m=1}^{n} \alpha_m \chi_{\Gamma_m} \; ; \; n \in \mathbb{Z}^+ \; \{\alpha_m\} \subseteq \mathbb{Q}, \; \Gamma_m \in \mathcal{C} \cup \{E\} \right\}.$$

Then S is dense in $L^1(\mu)$.

$\therefore$ If $\mathcal{C}$ is countable ~~&~~ $L^1(\mu)$ is separable.

Pf: Let $\overline{S}$ = closure of S in $L^1(\mu)$.

$\overline{S}$ is a vector space ($\mathbb{R}$).

So if $f \in L^1(\mu)$ and $f^+, f^- \in \overline{S}$ then $f \in \overline{S}$.

It suffices to show then that

any non-neg. $f \in L^1(\mu)$ $(f \geq 0)$

is contained in $\overline{S}$.

$f \geq 0$ is the monotone limit

of $\uparrow$ non-neg. simple functions. and $\therefore$ the

L^1 limit of simple fcns.

So it is enough to show that any

non-neg. simple fcn is in $\overline{S}$.

Since $\overline{S}$ is a vector space, it is

enough to show that $\chi_\Gamma \in \overline{S}$

for any $\Gamma \in \mathcal{B}$.

Consider $\mathcal{H} = \{ \Gamma \in \mathcal{B}$ s.t. $\chi_\Gamma \in \overline{S} \}$.

then $\mathcal{C} \subset \mathcal{H}$. and $E \in \mathcal{H}$.

and if $\Gamma_1, \Gamma_2 \in \mathcal{H}$ w/ $\Gamma_1 \cap \Gamma_2 = \phi$.

$$\chi_{\Gamma_1 \cup \Gamma_2} = \chi_{\Gamma_1} + \chi_{\Gamma_2} \in \overline{S}$$

so $\Gamma_1 \cup \Gamma_2 \in \mathcal{H}$.

if $\Gamma_1, \Gamma_2 \in \mathcal{H}$ s.t. $\Gamma_1 \subseteq \Gamma_2$ ~~all~~ then

$$\chi_{\Gamma_2 \setminus \Gamma_1} = \chi_{\Gamma_1} - \chi_{\Gamma_2} \in \overline{S}$$

so $\Gamma_2 \setminus \Gamma_1 \in \mathcal{H}$.

if $\{\Gamma_n\} \subset \mathcal{H}$ and $\Gamma_n \nearrow \Gamma$ then.

χ_Γ is a monotone limit of χ_{Γ_n}

$\therefore$ an L^1 limit so $\chi_\Gamma \in \overline{S}$.

and $\Gamma \in \mathcal{H}$.

$\therefore$ $\mathcal{H}$ is a d-system containing $\mathcal{E}$.

and $\therefore \mathcal{B} \subset \mathcal{H}$. $\blacksquare$

<u>Corollary</u>: (E, ρ) a metric space.

μ a measure on $(E, \mathcal{B}_E)$.

$\exists\ E_n$ open $\mu(E_n) < +\infty$.

$$E_n \subseteq E_{n+1} \quad \forall n \geq 1.$$

$$\bigcup_{n=1}^{\infty} E_n = E.$$

$$K_n = \{\varphi : \varphi \text{ is bdd, } \rho\text{-unif. cont, } \varphi(x) = 0\ \forall x \notin E_n\}$$

Then, $K = \bigcup_{n \geq 1} K_n$ is dense in $L^1(\mu)$.

<u>Pf:</u> ~~[illegible]~~

If $f \in L^1(\mu)$ then

$$\| f \chi_{E_n} - f \|_{L^1(\mu)} \to 0 \quad n \to \infty.$$

since $|(f\chi_{E_n} - f)| \leq |f|$ $\forall n$. (D.C.T)

So it suffices to show that

$f\chi_{E_n} \in \overline{K}$ for each n.

By the previous corollary, it is enough to show that

$\chi_G \in \overline{K_n}$ for any open $G \subset E_n$.

but

$$\varphi_m(x) = \left(\frac{\rho(x, G^c)}{1 + \rho(x, G^c)} \right)^{1/m}$$

is ρ- unif cont. and

$0 \leq \varphi_m(x) \nearrow \chi_G$ as $m \to \infty$.

$\therefore \varphi_m(x) \to \chi_G$ in L^1

($\varphi_m \equiv 0$ and off K_n) $\blacksquare$