Math 137 10-15-08

Weak Law of Large Numbers.

Recall: $\{X_i : i \in \mathcal{I}\}$ is a uniformly integrable collection of RV's iff.

$$\lim_{R \nearrow \infty} \sup_{i \in \mathcal{I}} E^P\left[|X_i|, |X_i| \geq R\right] = 0.$$

i.e. given $\epsilon > 0$

$\exists R > 0$ s.t $\forall i$

$$E^P\left[|X_i|, |X_i| \geq R\right] \leq \epsilon.$$

e.g. $\Omega = [0,1]$ $\quad dP = dx$

$$X_i(\omega) = \frac{1}{|x - a_i|^{1/2}} \qquad \{a_i\}_{i=1}^{\infty} = \mathbb{Q} \cap [0,1].$$

$0 \leq x \leq 1$

$$E^P\left[X_i^2\right] = +\infty \quad \forall i$$

but

$\forall i$ $$E^P\left[|X_i|, |X_i| \geq R\right] \leq 2 \int_0^{1/R^2} \frac{1}{\sqrt{x}}\, dx = \frac{4}{R}.$$

Thm: Let $\{X_n : n \in \mathbb{Z}^+\}$ be a uniformly P-integrable sequence of P-independent R.V's.

Then $$\frac{1}{n} \sum_{i=1}^{n} (X_i - E^P[X_i]) \to 0$$ in $L^1(P)$.

$\therefore$ in P-probability.

Pf: W.L.O.G. W.M.A. $E^P[X_n] = 0 \ \forall n \in \mathbb{Z}^+$

$0 < R < \infty$ let: $f_R(t) = t\,\chi_{[-R,R]}(t)$

$$m_n^{(R)} = E^P[f_R \circ X_n]$$

$$X_n^{(R)} = f_R \circ X_n - m_n^{(R)}$$

$$Y_n^{(R)} = X_n - X_n^{(R)}$$

$$\bar{S}_n^{(R)} = \frac{1}{n} \sum_{\ell=1}^{n} X_\ell^{(R)}$$

$$\bar{T}_n^{(R)} = \frac{1}{n} \sum_{\ell=1}^{n} Y_\ell^{(R)}$$

Now,

$$E[X_n] = 0 \implies m_n^{(R)} = -E[X_n, |X_n| > R].$$

so we have.

$$E^P[|\bar{S}_n|] \le E^P[|\bar{S}_n^{(R)}|] + E[|\bar{T}_n^{(R)}|]$$

$$\le E^P[|\bar{S}_n^{(R)}|^2]^{1/2} + \frac{1}{n}\sum_{\ell=1}^{n} E^P[|X_n - X_n^{(R)}|]$$

$$\le E^P[|\bar{S}_n^{(R)}|^2]^{1/2} + \frac{1}{n}\sum_{\ell=1}^{n} E^P[|X_n - f_R \circ X_n|] + |m_n^{(R)}|$$

$$\le E^P[|\bar{S}_n^{(R)}|^2]^{1/2} + 2\max_{1\le \ell \le n} E^P[|X_n|, |X_n| \ge R].$$

$$\le \left(\frac{1}{n^2}\sum_{\ell=1}^{n} E^P(X_\ell^{(R)2})\right)^{1/2} + 2\max_{1\le \ell \le n} E^P[|X_n|, |X_n| \ge R].$$

$$\le \frac{R}{\sqrt{n}} + 2\max_{1\le \ell \le n} E^P[|X_n|, |X_n| \ge R]$$

$\therefore$ for each $R > 0$

$$\overline{\lim_{n\to\infty}} E^P[|S_n|] \le 2 \sup E^P[|X_n|, |X_n| \ge R]$$

so $\overline{\lim_{n\to\infty}} E^P[|S_n|] = 0.$ $\to 0$ $R \to \infty$.

Theorem: Let $\{X_n : n \in \mathbb{Z}^+\}$ be a sequence of P-independent, integrable R.V.'s with $E(X_j) = 0$ $\forall j$. and

$$M_4 = \sup_{n \in \mathbb{Z}^+} E^P[X_n^4] < \infty.$$

Then for each $\epsilon > 0$

$$\epsilon^4 P\left(|\bar{S}_n| \geq \epsilon \right) \leq E^P[\bar{S}_n^4] \leq \frac{3M_4}{n^2}$$

$n \in \mathbb{Z}^+$.

and this implies that $\bar{S}_n \to 0$ P-a.e.

Pf: $$P\left(|\bar{S}_n| \geq \epsilon \right) \leq \int_{\{|\bar{S}_n| \geq \epsilon\}} \frac{|\bar{S}_n|^4}{\epsilon^4} \, dP \leq \frac{1}{\epsilon^4} E^P[\bar{S}_n^4].$$

Now

$$\left(\cancel{}S_{n+1} \right)^4 = S_n^4 + 4 S_n^3 X_{n+1} + 6 S_n^2 X_{n+1}^2 + 4 S_n X_{n+1}^3 + X_{n+1}^4$$

So $E(S_{n+1}^4) = E(S_n^4) + 6E(S_n^2)E(X_{n+1}^2)$

$$+ E(X_{n+1}^4)$$

and we assume inductively that

$$E(S_n^4) \le 3M_4 n^2.$$

Then

$$\begin{aligned}
E(S_{n+1}^4) &= E(S_n^4) + 6\sum_{j=1}^{n} E(X_j^2)\,E(X_{n+1}^2) \\
&\qquad + E(X_{n+1}^4)^{*} \\
&\le 3M_4 n^2 + 6\sum_{j=1}^{n} E(X_j^4)^{1/2} E(X_{n+1}^4)^{1/2} \\
&\qquad + E(X_{n+1}^4)^{*} \\
&\le 3M_4 n^2 + 6nM_4 + M_4. \\
&\le 3M_4(n^2 + 2n + 1/3) \\
&\le 3M_4(n+1)^2.
\end{aligned}$$

and we have the inequality for all n.

Now $\displaystyle\sum_{n=1}^{\infty} P(|\bar{S}_n| \geq \epsilon) \leq \frac{3M_4}{\epsilon^4}\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty$.

and Borel-Cantelli

$$\Rightarrow \quad P\left(\overline{\lim_{n\to\infty}}\left[\{|\bar{S}_n| \geq \epsilon\}\right]\right) = 0.$$

$$\therefore \quad P\left(\overline{\lim_{n\to\infty}}\ |S_n| \geq \epsilon\right) = 0.$$

~~Theorem~~

<u>Lemma</u>: For any sequences

$\{a_n : n \in \mathbb{Z}^+\}$, $\{b_n : n \in \mathbb{Z}^+\}$

s.t. b_n converges to an el. of $(0,\infty]$

the set

$$\left\{x : \lim_{n\to\infty} \frac{S_n - a_n}{b_n} \text{ exists in } \mathbb{R}\right\}$$

has P measure 0 or 1.

if $\quad b_n \to \infty$ as $n \to \infty$

then

$$\overline{\lim_{n\to\infty}} \frac{S_n - a_n}{b_n} \quad \text{and} \quad \underline{\lim_{n\to\infty}} \frac{S_n - a_n}{b_n}$$

are constant P a.e.

<u>Pf:</u> $\left\{ \lim_{n\to\infty} \frac{S_n - a_n}{b_n} \text{ exists in } \mathbb{R} \right\}$

$$= \left\{ \lim_{m\to\infty} \sup_{n \geq m} \left| \frac{S_{n+m} - a_{n+m}}{b_{n+m}} - \frac{S_m - a_m}{b_m} \right| = 0 \right\}$$

is a tail event. in the tail generated by $\{\sigma(X_n)\}_{n=1}^{\infty}$

~~Also, for any c and $\epsilon > 0$~~

$\cancel{\left\{ c \leq \overline{\lim_{n\to\infty}} \frac{S_n - a_n}{b_n} \leq c + \epsilon \right\}}$ ~~is a tail~~

Also, (if $b_n \to \infty$).

$$\overline{\lim_{n\to\infty}}\left[\frac{S_{n+m}-S_m-a_{n+m}}{b_{m+n}}\right] = \overline{\lim}\left(\frac{S_n-a_n}{b_n}\right)$$

$\boxed{\text{if } b_n \to \infty.}$

$$\underline{\lim_{n\to\infty}}\left(\frac{S_{n+m}-S_m-a_{n+m}}{b_{m+n}}\right) = \underline{\lim}\,\frac{S_n-a_n}{b_n}$$

So for any c and $\epsilon > 0$.
and m_0

$$\left\{ c < \overline{\lim}\left[\frac{S_n-a_n}{b_n}\right] < c+\epsilon \right\}$$

$$= \bigcap_{m\ge m_0} \left\{ c < \left[\overline{\lim_{n\to\infty}}\,\frac{S_{n+m}-S_m-a_{n+m}}{b_{m+n}}\right] < c+\epsilon \right\}.$$

and $\therefore$ is a tail event.

Similarly for $\underline{\lim}$ (if $b_n \to \infty$).

Theorem: If the X_n's are independent, square P-integrable and

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$$

then

$$\sum_{n=1}^{\infty} (X_n - E^P[X_n])$$

converges P-almost surely.

Note that

$$\sup_{n \geq N} P\left(\left| \sum_{l=N}^{n} (X_l - E^P[X_l]) \right| \geq \epsilon \right)$$

$$\leq \frac{1}{\epsilon^2} \sum_{l=N}^{\infty} \mathrm{Var}(X_l)$$

$\left(P(|\quad| \geq \epsilon) \leq \frac{1}{\epsilon^2} \int |\quad|^2 dP \right.$ and cross terms are zero by independence $\left.\right)$

So our Cauchy-Criteria for convergence in measure shows that $\sum_{n=1}^{\infty}(X_n - E^P[X_n])$ converges in P-probability

If we can bring the "$\sup_{n \geq N}$" inside the $P(\quad)$ then we will have our Cauchy Criteria for a.e. convergence.

<u>Theorem:</u> (Kolmogorov's inequality)

If X_n's are independent and square P integrable then

$$P\left(\sup_{n \geq 1}\left|\sum_{\ell=1}^{n}(X_\ell - E^P[X_\ell])\right| \geq \epsilon\right) \leq \frac{1}{\epsilon^2}\sum_{n=1}^{\infty} Var(X_n)$$

Pf: WLOG WMA $E(X_n)=0 \quad \forall n.$

If $1 \le n < N$ then

$$S_N^2 - S_n^2 = (S_N - S_n)^2 + 2(S_N - S_n)S_n$$

$$\ge 2(S_N - S_n)S_n.$$

Now $S_N - S_n$ has $E(S_N - S_n) = 0$

and is independent of $\sigma(X_1, \dots, X_n)$

So if $A_n \in \sigma(X_1, \dots, X_n)$

then.

$$E((S_N^2 - S_n^2)\chi_{A_n}) \ge E(2(S_N - S_n)S_n \chi_{A_n})$$

$$= 0.$$

and $\therefore$

$$E(S_N^2, A_n) \ge E(S_n^2, A_n).$$

for any $A_n \in \sigma(X_1, \dots, X_n)$.

Let $A_1 = \{ |S_1| > \epsilon \}$

$\vdots$

$A_{n+1} = \{ |S_{n+1}| > \epsilon \text{ and } \max_{1 \le l \le n} |S_l| \le \epsilon \}$

$n \in \mathbb{Z}^+$.

then

$A_m \cap A_n = \emptyset \qquad m \neq n.$

and

$$B_N \equiv \left\{ \max_{1 \le n \le N} |S_n| > \epsilon \right\} = \bigcup_{n=1}^{N} A_n$$

so

$$E^P[S_N^2, B_N] = \sum_{n=1}^{N} E^P[S_N^2, A_n]$$

$$\ge \sum_{n=1}^{N} E^P[S_n^2, A_n]$$

$$\ge \epsilon^2 \sum_{n=1}^{N} P(A_n) = \epsilon^2 P(B_N).$$

and in particular

$$\epsilon^2 P\left(\sup_{n \ge 1} |S_n| > \epsilon \right) = \lim_{N \to \infty} \epsilon^2 P(B_N)$$

$$\le \lim_{N \to \infty} E^P[S_N^2] \le \sum_{n=1}^{\infty} E^P[X_n^2]$$

as $\epsilon \nearrow \epsilon_0$.

$$\left(\sup_{n \geq 1} |S_n| > \epsilon \right) \searrow \left(\sup_{n \geq 1} |S_n| \geq \epsilon_0 \right).$$

So by monotonicity of measures (P)

$$\epsilon^2 P\left(\sup_{n \geq 1} |S_n| \geq \epsilon \right) \leq \sum_{n=1}^{\infty} E^P(X_n^2).$$

$\blacksquare$

To prove the 1st theorem

w.m.a. that $E^P(X_n) = 0$.

and apply last result to $\{X_{N+n} : n \in \mathbb{Z}^+\}$

to get.

$$P\left(\sup_{n > N} |S_n - S_N| \geq \epsilon \right)$$

$$\leq \frac{1}{\epsilon^2} \sum_{n=N+1}^{\infty} E^P[X_n^2]$$

Since $\sum_{n=1}^{\infty} Var(X_n) = \sum_{n=1}^{\infty} E(X_n^2) < +\infty$.

we have.

$$P\left(\sup_{n > N} |S_n - S_N| \geq \epsilon \right) \longrightarrow 0 \quad N \to \infty$$

which is our P a.e. Cauchy Criteria.

Integration by parts for Stieltjes Integrals

$$\int_a^b f(x)\,dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,df(x).$$

<u>Summation by Parts:</u> $\{a_j\}_{j=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{k=M}^{N-1} B_k (a_{k+1} - a_k)$$

where $B_0 = 0$

$$B_k = \sum_{j=1}^{k} b_j$$

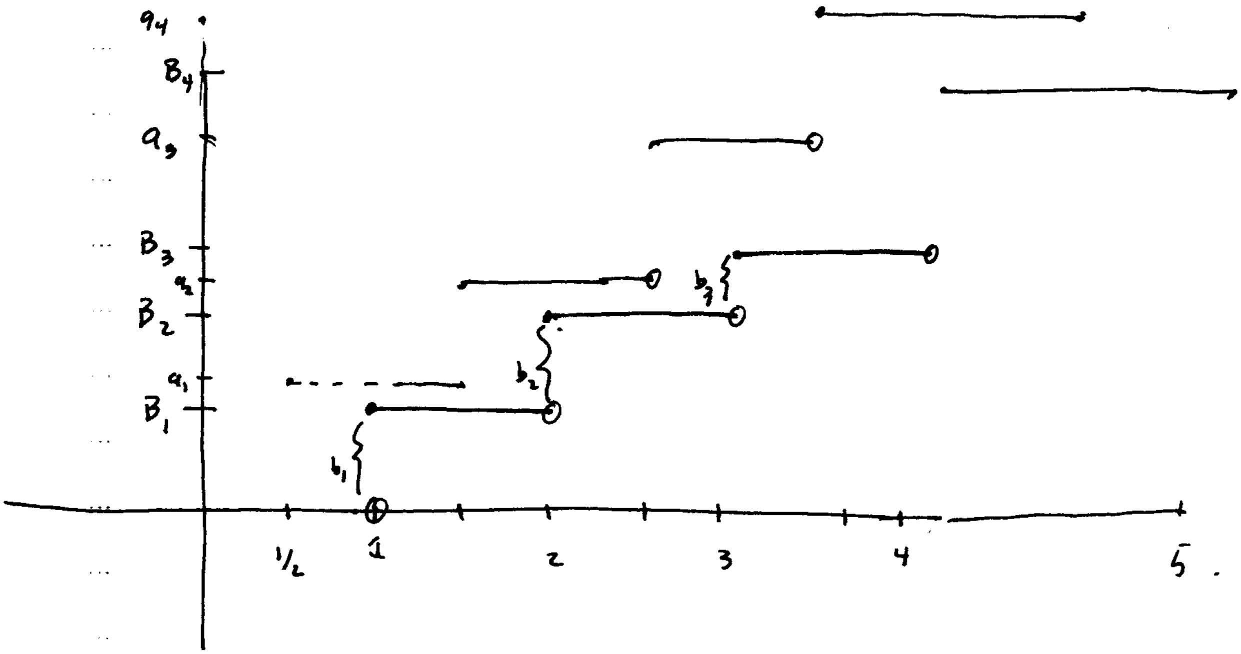
$$\sum_k a_k b_k = \int a(t)\,dB(t)$$ a, B step functions

$B(t)$ jumps from B_{k-1} to B_k when $t=k$.

$a(t)$ jumps by $a_{k+1} - a_k$ at $t = k + 1/2$.

then

$$\sum_{n=m}^{N} a_n b_n = \int_{m^-}^{N^+} a(t)\, dB(t).$$



and ∴

$$\sum_{n=m}^{N} a_n b_n = a(t)B(t)\Big|_{m^-}^{N^+} - \int_{m^-}^{N^+} B(t)\, da(t)$$

$$= a_N B(N) - a_m B_{m-1}$$

$$- \sum_{n=m}^{N-1} B_n (a_{n+1} - a_n)$$