Math 137 10-13-08.

Theorem: $\{f_n\}_{n=1}^{\infty}$ $\mathbb{R}$ valued, msble. on $(E, \mathcal{B}, \mu)$.

T.F.A.E.

i) $\exists$ $\mathbb{R}$ vld. msble f s.t.

$$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f - f_n| > \epsilon \right) = 0 \quad \forall \epsilon > 0.$$

ii) $$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_m - f_n| > \epsilon \right) = 0 \quad \forall \epsilon > 0.$$

moreover,

i) $\Rightarrow$ $f_n \to f$ both a.e and in μ measure and (finally)

if $\mu(E) < \infty$ then $f_n \to f$ a.e iff i) holds.

So on a finite measure space μ-a.e. convergence $\Rightarrow$ convergence in measure.

Recall from ~~last time~~ (2 times ago).

- $f_n \xrightarrow{\mu} f$ iff for any $\epsilon > 0$

$$\mu\{|f_n - f| > \epsilon\} \to 0 \quad \text{as } n \to \infty.$$

- We proved that ii) $\Rightarrow$ f_n converges a.e., Naming the a.e. limit f we found that i) holds for it.

Now i) implies that $f_n \to f$ a.e. and $f_n \xrightarrow{\mu} f$

given i),
That $f_n \xrightarrow{\mu} f$ is clear.

To see that $f_n \to f$ a.e.:

For each $\epsilon > 0$,

$$\bigcap_{m \geq 1} \{\sup_{n \geq m} |f - f_n| \geq \epsilon\}$$ is the limit

of the decreasing sequence of sets.

$$\{\sup_{n \geq m} |f - f_n| \geq \epsilon\} \qquad m = 1, 2, \ldots$$

since their measures tend to zero we have monotonicity and.

$\therefore \mu\left(\bigcap_m \{ \sup_{n \geq m} |f-f_n| \geq \epsilon \} \right) = 0.$

Let $\epsilon_k \searrow 0$ (e.g. $\epsilon_k = 1/k$).

then $A = \bigcup_{k=1}^{\infty} \bigcap_m \{ \sup_{n \geq m} |f-f_n| \geq \epsilon_k \}$

has $|A| = 0$.

If $x \in A^c$ then

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{ \sup_{n \geq m} |f-f_n| < \epsilon_k \}.$$

Given $\epsilon > 0$ choose k' s.t.

$\epsilon_{k'} < \epsilon$, then

$$x \in \bigcup_{m=1}^{\infty} \{ \sup_{n \geq m} |f-f_n| < \epsilon_{k'} \}.$$

So $\exists\, m$ s.t.

$$\sup_{n \geq m} |f(x) - f_n(x)| < \epsilon_{k'} < \epsilon.$$

i.e. $f_n(x) \to f(x)$.

If $\mu(E) < \infty$ and $f_n \to f$. a.e.

then since $f_n \to f$ a.e.

$$\mu\left(\bigcap_{m=1}^{\infty} \{\sup_{n\geq m} |f_n - f| \geq \epsilon\}\right) = 0$$

and. (since $\mu(E) < +\infty$).

$$\lim_{m\to\infty} \mu\left(\{\sup_{n\geq m} |f_n - f| \geq \epsilon\right)$$

$$= \mu\left(\bigcap_{m=1}^{\infty} \{\sup_{n\geq m} |f_n - f| \geq \epsilon\right) = 0.$$

So i) holds.

The previous theorem gives a Cauchy Criteria for μ a.e. convergence.

The next theorem gives one for convergence in measure.

Theorem: $\{f_n\}_{n=1}^{\infty}$ $\mathbb{R}$ valued msble.

in $(E, \mathcal{B}, \mu)$

$\exists$ f $\mathbb{R}$ valued msble s.t.

$$f_n \xrightarrow{\mu} f$$

iff $\lim_{m \to \infty} \sup_{n \geq m} \mu(|f_n - f_m| \geq \epsilon) = 0$ $\forall\, \epsilon > 0$

Further, if $f_n \xrightarrow{\mu} f$, then $\exists$ a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ s.t.

$\lim_{i \to \infty} \mu(\sup_{j \geq i} |f - f_{n_j}| \geq \epsilon) = 0$

$\epsilon > 0$.

$\therefore$ $f_{n_i} \to f$ a.e. (and in measure)

Pf: If $f_n \xrightarrow{\mu} f$ then, given $\epsilon > 0$.

$$\mu\{|f_n - f_m| \geq \epsilon\} \leq \mu\{|f_n - f| \geq \epsilon/2\} + \mu\{|f_m - f| \geq \epsilon/2\}$$

so $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \epsilon) = 0$.

Conversely, if

$$\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \epsilon) = 0.$$

Choose n_i

$1 \leq n_1 < n_2 < \dots$ s.t.

$$\sup_{n \geq n_i} \mu(|f_n - f_{n_i}| \geq 2^{-(i+1)}) \leq 2^{-(i+1)}$$

$i \geq 1$.

Then since (for $k > i$).

$$f_{n_k} - f_{n_i} = (f_{n_k} - f_{n_{k-1}}) + (f_{n_{k-1}} - f_{n_{k-2}}) + \dots$$

$$+ \dots (f_{n_{i+2}} - f_{n_{i+1}}) + (f_{n_{i+1}} - f_{n_i}),$$

if $|f_{n_j} - f_{n_{j-1}}| < 2^{-j}$ for each j.

we have

$$|f_{n_n} - f_{n_i}| \le 2^{-n} + 2^{-(n+1)} + \dots + 2^{-(i+1)}$$

$$= \sum_{j=i+1}^{n} 2^{-j} = 2^{-(i+1)} \sum_{j=0}^{n-(i+1)} 2^{-j}$$

$$= 2^{-(i+1)} \frac{1 - 2^{-(n-i)}}{1 - 1/2}$$

$$= 2^{-i}\left(1 - 2^{-(n-i)}\right) < 2^{-i}.$$

So,

$$\mu\left(\sup_{j \ge i} |f_{n_j} - f_{n_i}| > 2^{-i} \right)$$

$$\le \mu\left(\bigcup_{j \ge i} \{ |f_{n_{j+1}} - f_{n_j}| \ge 2^{-(j+1)} \} \right).$$

$$\le \sum_{j \ge i} 2^{-(j+1)} = 2^{-(i+1)} \sum_{j=0}^{\infty} 2^{-j} = 2^{-i}.$$

$\therefore$ f_{n_i} satisfies the a.e Cauchy criteria from the previous theorem.

and there is an f s.t.

$f_{n_i} \to f$ μ a.e. and

$f_{n_i} \xrightarrow{\mu} f$.

For each i, and each m, we have

$$\mu\{|f_m - f| \geq \epsilon\} \leq \mu\{|f_m - f_{n_i}| \geq \epsilon/2\} + \mu(|f_{n_i} - f| \geq \epsilon/2).$$

and $\therefore$

$$\mu\{|f_m - f| \geq \epsilon\} \leq \overline{\lim_{i\to\infty}}\, \mu\{|f_m - f_{n_i}| \geq \epsilon/2\} + \underbrace{\overline{\lim_{i\to\infty}}\, \mu\{|f_{n_i} - f| \geq \epsilon/2\}}_{=0}.$$

$$\leq \sup_{n \geq m} \mu\left(|f_n - f_m| \geq \frac{\epsilon}{2}\right) \to 0. \quad m \to \infty.$$

Note that f_{n_i} converges a.e. and in measure (to the same limit).

Weak Law of Large Numbers

$(\Omega, \mathcal{F}, P)$ a probability space

$\{X_n\}$ a sequence of real vld. RV's on $(\Omega, \mathcal{F}, P)$

For $n \in \mathbb{Z}^+$:

$$S_n = X_1 + \dots + X_n$$

$$\bar{S}_n = \frac{1}{n}(X_1 + \dots + X_n).$$

Lemma: $E^P[X_n^2] < \infty \quad \forall n \in \mathbb{Z}^+$

$E^P[X_k X_\ell] = 0 \quad k \neq \ell.$

$\Rightarrow$

For each $\epsilon > 0$.

$$\epsilon^2 P(|\bar{S}_n| \geq \epsilon) \leq E^P[\bar{S}_n^2] = \frac{1}{n^2} \sum_{\ell=1}^{n} E^P[X_\ell^2].$$

$$\forall n \in \mathbb{Z}^+.$$

Pf: $E^P[S_n^2] = E^P\left(\sum_{i,j} X_i X_j\right)$ $1 \le i \le n$, $1 \le j \le n$

$$= E^P\left(\sum_{i=1}^{n} X_i^2\right). = \sum_{i=1}^{n} E^P(X_i^2).$$

We have., for any R.V. Y

$$\epsilon^2 \chi_{[\epsilon,\infty)}(|Y|) \le Y^2 \chi_{[\epsilon,\infty)}(|Y|) \le Y^2.$$

and integrating this gives

$$\epsilon^2 P(|Y| \ge \epsilon) \le \int_{\{|Y|\ge\epsilon\}} Y^2 dP \le \int_{\Omega} Y^2 dP.$$

So,

$$E^P[\bar{S}_n^2] = \frac{1}{n^2}\sum_{i=1}^{n} E^P[X_i^2] \ge \epsilon^2 P(|\bar{S}_n| \ge \epsilon).$$

Corollary: if $M \equiv \sup_{n\in\mathbb{Z}^+} E^P[X_n^2] < \infty$.

then

$$\epsilon^2 P(|\bar{S}_n| \ge \epsilon) \le E^P[\bar{S}_n^2] \le \frac{M}{n}.$$

So $|\bar{S}_n| \xrightarrow{P} 0$ and $\bar{S}_n \xrightarrow{L^2(P)} 0$

$$\mathrm{Var}(X) \equiv E^P\left(\left(X - E^P[X]\right)^2\right) =$$

$$E^P[X^2] - \left(E^P[X]\right)^2 \le E^P[X^2].$$

So,

If X_n are P independent

and $E(X_n^2) < +\infty \quad \forall n \in \mathbb{Z}^+$.

then $E^P\left(\left(X_n - E^P(X_n)\right)^2\right) < +\infty \quad \forall n \in \mathbb{Z}$.

and $E\Big(\big(X_i - E^P(X_i)\big)\big(X_j - E(X_j)\big)\Big) = 0 \quad i \neq j$.

(We then have).

<u>Theorem:</u>

if
$$\begin{cases} \{X_n\}_{n=1}^{\infty} & P\text{-independent} \\ E(X_n) = m. & \\ \mathrm{Var}(X_n) \le \sigma^2 \ \forall n. & \end{cases}$$

(so that $E[X_n^2] \le \sigma^2 + m^2 \quad \forall n$).

then (by our lemma)

$$\epsilon^2 P(|\bar{S}_n - m| \geq \epsilon) \leq E^P[(\bar{S}_n - m)^2]$$
$$\leq \frac{\sigma^2}{n}.$$

So $\bar{S}_n \to m$ in $L^2(P)$

and ($\therefore$) in P-probability.

Using independence, we can weaken the conditions required of the X_n

reason: If the X_n are ind. then so are

$$X_n \cdot \chi_{[-R,R]}(X_n)$$

for any $R > 0$.