Math 137 10/8/08

<u>Theorem</u>: $\{f_n\}_{n=1}^{\infty}$ $\mathbb{R}$ valued, msble on $(E, \mathcal{B}, \mu)$

T.F.A.E.

i) $\exists$ $\mathbb{R}$ valued msble f s.t.

$$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f-f_n| \geq \epsilon\right) = 0, \quad \epsilon > 0$$

ii) $$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_n - f_m| \geq \epsilon\right) = 0, \quad \epsilon > 0$$

moreover,

i) $\Rightarrow$ $f_n \to f$ both a.e. and in μ measure

and (finally)

if $\mu(E) < \infty$ then ~~and~~ $f_n \to f$ a.e. iff (i) holds.

so on a finite measure space

μ-a.e. convergence $\Rightarrow$ convergence in measure.

Recall from last time

- $f_n \xrightarrow{\mu} f$ iff for any $\epsilon > 0$

$$\mu\{|f_n - f| > \epsilon\} \to 0 \quad \text{as } n \to \infty.$$

- We proved that (ii) $\Rightarrow$ f_n converges a.e. Naming the a.e. limit f we found that (i) holds for it

- i) easily implies that $f_n \to f$ a.e and $f_n \xrightarrow{\mu} f$.

For each $\epsilon > 0$, $\bigcap_m \{\sup_{n \geq m} |f - f_n| \geq \epsilon\}$ is the decreasing limit of the sets $\{\sup_{n\geq m} |f - f_n| \geq \epsilon\}$.

(since i) $\Rightarrow$ these measures tend to zero we have monotonicity).

$\therefore$ has measure zero.

Let $\epsilon_k \searrow 0$ (e.g. $\epsilon_k = 1/k$)

then $A = \bigcup_{k=1}^{\infty} \bigcap_m \{\sup_{n \geq m} |f - f_n| \geq \epsilon_k\}$

has measure zero.

if $x \in A^c$ then $x \in \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{\sup_{n\geq m} |f - f_n| < \epsilon_k\}$.

given $\epsilon > 0$, choose k' s.t. $\epsilon_{k'} < \epsilon$ then $x \in \bigcup_{m=1}^{\infty} \{\sup_{n \geq m} |f - f_n| < \epsilon_{k'}\}$.

So $\exists\, m$ s.t.

$$\sup_{n \geq m} |f(x) - f_n(x)| < \epsilon_{n'} < \epsilon .$$

i.e. $f_n(x) \cancel{\Rightarrow} f(x)$.

And The convergence in measure is clear.

If $\mu(E) < \infty$ and $f_n \to f$ a.e.

then (since $f_n \to f$) a.e. $\mu\left(\bigcap_{m=1}^{\infty} \{ \sup_{n \geq m} |f_n - f| \geq \epsilon \} \right) = 0.$

and since $\mu(E) < \infty$.

$$\lim_{m \to \infty} \mu\left(\sup_{n \geq m} |f_n - f| \geq \epsilon \right) = \mu\left(\bigcap_{m=1}^{\infty} \{ \sup_{n \geq m} |f_n - f| \geq \epsilon \} \right) = 0$$

so i) holds.

The previous theorem gives a Cauchy criteria for μ a.e. convergence.

The next theorem gives one for convergence in measure.

<u>Theorem</u>: $\{f_n\}_{n=1}^{\infty}$ $\mathbb{R}$ valued mble

on $(E, \mathcal{B}, \mu)$.

$\exists$ f $\mathbb{R}$ valued mble s.t.

$$f_n \xrightarrow{\mu} f$$

iff $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \epsilon) = 0$ $\epsilon > 0$.

Further,

if $f_n \xrightarrow{\mu} f$, then $\exists$ a subsequence

$\{f_{n_j}\}_{j=1}^{\infty}$ s.t.

$$\lim_{i\to\infty} \mu\left(\sup_{j\geq i} |f - f_{n_j}| \geq \epsilon\right) = 0 \quad \epsilon > 0.$$

$\therefore$ $f_{n_i} \to f$ a.e. (and in measure).

Pf. If $f_n \xrightarrow{\mu} f$ then.

$$\mu\{|f_n - f_m| \geq \epsilon\} \leq \mu\{|f_n - f| \geq \tfrac{\epsilon}{2}\} + \mu\{|f_m - f| \geq \tfrac{\epsilon}{2}\}$$

so $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \epsilon) = 0$.

Conversely,

if $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \epsilon) = 0$.

Choose $1 \leq n_1 < n_2 < \ldots < n_i < \ldots <$ ~~...~~

so that

$$\sup_{n \geq n_i} \mu(|f_n - f_{n_i}| \geq 2^{-(i+1)}) \leq 2^{-(i+1)} \qquad i \geq 1.$$

Then

$$\mu\left(\sup_{j\geq i} |f_{n_j} - f_{n_i}| > 2^{-i}\right)$$

$$\leq \mu\left(\bigcup_{j\geq i} \{|f_{n_{j+1}} - f_{n_j}| \geq 2^{-(j+1)}\}\right)$$

(Since

$$f_{n_k} - f_{n_i} = (f_{n_k} - f_{n_{k-1}}) + (f_{n_{k-1}} - f_{n_{k-2}})$$

$$\cdots + (f_{n_{i+2}} - f_{n_{i+1}}) + (f_{n_{i+1}} - f_{n_i})$$

then if $|f_{n_j} - f_{n_{j-1}}| < 2^{-j}$ for each j.

we have.

$$|f_{n_k} - f_{n_i}| \le 2^{-k} + 2^{-(k-1)} + \ldots 2^{-(i+1)}.$$

$$= \sum_{j=i+1}^{k} 2^{-j} = 2^{-(i+1)} \sum_{j=0}^{k-(i+1)} 2^{-j}$$

$$= 2^{-(i+1)} \frac{1 - 2^{-(k-i)}}{1 - 1/2}$$

$$= 2^{-i}(1 - 2^{-(k-i)}) < 2^{-i}).$$

and

$$\mu \left(\bigcup_{j \ge i} \{ |f_{n_{j+1}} - f_{n_j}| \ge 2^{-(j+1)} \} \right)$$

$$\le \sum_{j \ge i} \mu \{ |f_{n_{j+1}} - f_{n_j}| \ge 2^{-(j+1)} \} \le \sum_{j \ge i} 2^{-(j+1)}$$

$$\le 2^{-(i+1)} \sum_{j=0}^{\infty} 2^{-j} = 2^{-i}.$$

$\therefore \{f_{n_i}\}_{i=1}^{\infty}$ satisfies the a.e. Cauchy criteria from the previous theorem and there is an f s.t.

$f_{n_i} \to f$ μ almost everywhere and

$f_{n_i} \xrightarrow{\mu} f$.

For each i, and each m we have.

$$\mu\{|f_m - f| \geq \epsilon\} \leq \mu\{|f_m - f_{n_i}| \geq \epsilon/2\}$$

$$+ \mu(|f_{n_i} - f| \geq \epsilon/2$$

and $\therefore$

$$\mu\{|f_m - f| \geq \epsilon\} \leq \overline{\lim_{i\to\infty}}\, \mu\{|f_m - f_{n_i}| \geq \epsilon/2\}$$

$$+ \underbrace{\overline{\lim_{i\to\infty}}\, \mu\{|f_{n_i} - f| \geq \epsilon/2\}}_{= 0}.$$

$$\leq \sup_{n \geq m} \mu\left(|f_n - f_m| \geq \frac{\epsilon}{2}\right) \to 0.$$

Note that f_{n_i} converges a.e. and in measure (and the limits are the same.)

<u>Independence of random variables</u>

~~E~~ $(\Omega, \mathcal{F}, P)$ a probability space.

$i \in I$, X_i a random variable on Ω
(msble function on Ω)

with values in (E_i, B_i)
(a msble space).

<u>Def</u>: The X_i are mutually P-independent if the σ-algebras

$$\sigma(X_i) = X_i^{-1}(\mathcal{B}_i) \equiv \{X_i^{-1}(B_i) : B_i \in \mathcal{B}_i\}$$

$$i \in I$$

are P-independent

<u>remark</u>: ($X_i^{-1}(\mathcal{B}_i)$ is the smallest sub σ-algebra (in $\mathcal{F}$) which makes X_i a msble fcn into (E_i, B_i)).

<u>remark</u>:
Quite often $(E_i, B_i) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. for each i.

Independence of the X_i, by definition means

$$E^P\left[(\chi_{B_{i_1}} \circ X_{i_1}) \cdots \chi_{B_{i_n}} \circ X_{i_n}\right]$$

$$= E^P(\chi_{B_{i_1}} \circ X_{i_1}) \cdots E^P(\chi_{B_{i_n}} \circ X_{i_n})$$

for any finite distinct choice $(i_1, \ldots, i_n)$ from $\mathcal{I}$.
and any sets $B_{i_j} \in \mathcal{B}_{i_j}$ $j = 1, \ldots, n$.

By linearity of the integral this is equivalent to

$$E^P\left[(f_{i_1} \circ X_{i_1}) \cdots f_{i_n} \circ X_{i_n}\right]$$

$$= E^P(f_{i_1} \circ X_{i_1}) \cdots E^P(f_{i_n} \circ X_{i_n})$$

for any finite distinct choice $i_1, \ldots, i_n$ from $\mathcal{I}$
and simple fcns f_{i_j}

with f_{i_j} msble in $\mathcal{B}_{i_j}$. $j = 1, \ldots, n$.

By dominated convergence this is equivalent to the same statement with

f_{i_j} bdd msble $\mathbb{R}$ valued on $(E_{i_j}, \mathcal{B}_{i_j})$.

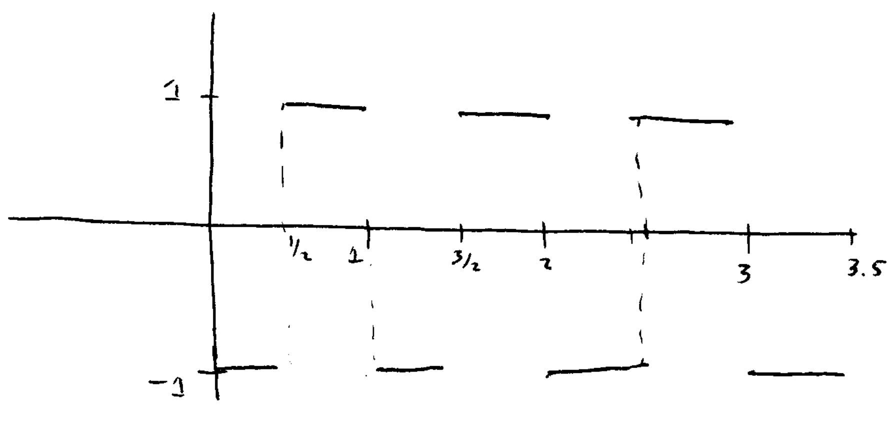
Take $(\Omega, \mathcal{F}) = \left([0,1), \mathcal{B}_{[0,1)}\right)$.

$P =$ Lebesgue measure on $[0,1)$.

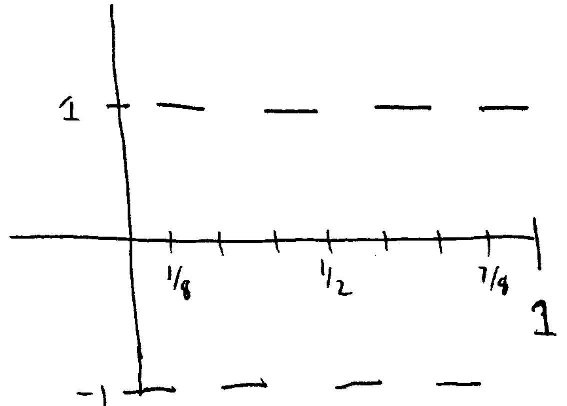
for $t \in \mathbb{R}$, $\lfloor t \rfloor = \sup\{n \in \mathbb{Z} : n \le t\}$.

$R : \mathbb{R} \to \{-1, 1\}$

$$R(t) = \begin{cases} -1 & \text{if } t - \lfloor t \rfloor \in [0, 1/2) \\ 1 & \text{if } t - \lfloor t \rfloor \in [1/2, 1) \end{cases}$$



$R_n(\omega) \equiv R(2^{n-1}\omega) \quad n \in \mathbb{Z}^+ \quad \omega \in [0,1)$.



$R_3(\omega)$. $0 \le \omega < 1$

<u>Claim:</u> The Rademacher Functions (R_n) are P-independent.

Any $\mathbb{R}$ valued function on $\{-1, 1\}$ is

$$\alpha + \beta x \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

So it is enough to show that

$$E^P\left[(\alpha_1 + \beta_1 R_1) \cdots (\alpha_n + \beta_n R_n)\right] = \alpha_1 \cdots \alpha_n.$$

for any n and any $(\alpha_j, \beta_j) \in \mathbb{R}^2$.

The case $n = 1$ is clear so we assume inductively that the desired conclusion holds for n.

Then $E^P\left[(\alpha_1 + \beta_1 R_1) \cdots (\alpha_n + \beta_n R_n)(\alpha_{n+1} + \beta_{n+1} R_{n+1})\right]$

$$= \alpha_{n+1} E^P\left[(\alpha_1 + \beta_1 R_1) \cdots (\alpha_n + \beta_n R_n)\right] + \beta_{n+1} E^P\left((\alpha_1 + \beta_1 R_1) \cdots (\alpha_n + \beta_n R_n) R_{n+1}\right)$$

$$= \alpha_{n+1} \alpha_n \cdots \alpha_1 + \beta_{n+1} E^P\left((\alpha_1 + \beta_1 R_1) \cdots (\alpha_n + \beta_n R_n) R_{n+1}\right)$$

So its enough to check that

$$(*)\quad E^P[F(R_1,\ldots,R_n)R_{n+1}] = 0.$$

for any $F:[\{-1,1\}]^n \to \mathbb{R}$.

But

$(R_1,\ldots,R_n)$ is constant on each

$$\left[\frac{m}{2^n}, \frac{m+1}{2^n}\right) \qquad 0 \le m < 2^n.$$

and $E^P(R_{n+1}\chi_{[\frac{m}{2^n},\frac{m+1}{2^n}]}) = 0$. $\quad 0 \le m < 2^n$.

$\therefore$ $(*)$ holds.

A random Variable U is uniformly distributed on $[a,b]$ if

$$\text{Prob}[U \le x] = \begin{cases} 0 & x \le a. \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x \geq b. \end{cases}$$

<u>Lemma</u>: Let $\{Y_\ell : \ell \in \mathbb{Z}^+\}$ be a sequence of P-independent $\{0,1\}$ valued random variables with $E[Y_\ell] = 1/2$ $\forall \ell$ on $(\Omega, \mathcal{F}, P)$ and let

$$U = \sum_{\ell=1}^{\infty} \frac{X_\ell}{2^\ell} .$$

Then U is uniform on $[0,1]$.

<u>Pf</u>: Since everything ~~to~~ be concluded depends only on the c.d.f.'s of the Y_ℓ and ~~the~~ U, we may assume $(\Omega, \mathcal{F}, P) = [[0,1), \mathcal{B}_{[0,1)}, \lambda_{[0,1)}]$ and $Y_\ell = \frac{R_\ell(\omega) + 1}{2} = \epsilon_\ell(\omega)$ ~~on~~ $(\Omega, \mathcal{F}, P)$

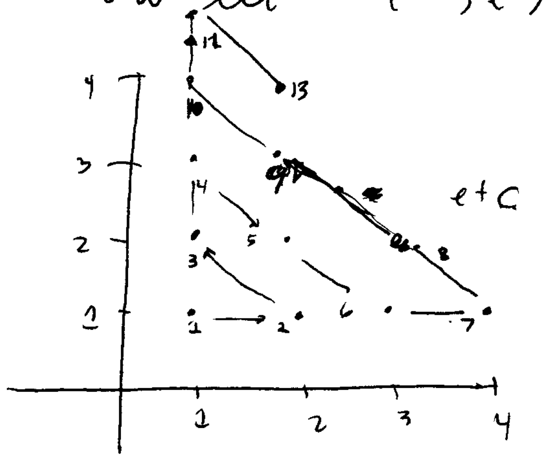
For each $\omega \in [0,1]$

$$\omega = \sum_{n=1}^{\infty} \epsilon_n(\omega) \cdot 2^{-n} \quad (\text{binary expansion of } \omega)$$

So the conclusion holds.

Now let $(k,\ell) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \to n(k,\ell)$.

be a 1-1 mapping of $\mathbb{Z}$ onto $\mathbb{Z}^+$



and put $$Y_{k,\ell} = \frac{1 + R_{n(k,\ell)}}{2}. \qquad (k,\ell) \in (\mathbb{Z}^+)^2$$

$Y_{k,\ell}$ is $\{0,1\}$ Bernoulli for each $k,\ell)$.

$\{Y_{k,\ell} : (k,\ell) \in (\mathbb{Z}^+)^2\}$ is P independent

$$\therefore \quad U_k \equiv \sum_{\ell=1}^{\infty} \frac{Y_{k,\ell}}{2^\ell} \qquad k \in \mathbb{Z}^+$$

is uniform on $[0,1]$.

and the U_k are mutually independent.

Given a c.d.f F on $\mathbb{R}$

i.e.

F is right continuous, non-decreasing

s.t $\lim_{x \to -\infty} F(x) = 0$ $\lim_{x \to +\infty} F(x) = 1$.

Let $F^{-1}(t) = \inf \{ s \in \mathbb{R} : F(s) \geq t \}$ $t \in [0,1]$

$(\inf \{ \emptyset \} = +\infty)$.

Then if U is uniformly dist. on $[0,1]$

$P(F^{-1} \circ U \leq t)$

$= P(\inf \{ s \in \mathbb{R} : F(s) \geq U \} \leq t)$

$= P(U \leq F(t)) = F(t)$

So $X = F^{-1} \circ U$ has the c.d.f F.

We have shown

<u>Theorem:</u> $\Omega = [0,1)$

$\mathcal{F} = \mathcal{B}_{[0,1)}$

$P = \lambda_{[0,1)}$

Then for any sequence $\{F_n\}_{n \in \mathbb{Z}^+}$

of c.d.f's. on $\mathbb{R}$

$\exists \ \{X_n : n \in \mathbb{Z}^+\}$

which are P-independent on $(\Omega, \mathcal{F}, P)$

and s.t.

$$P(X_n \le t) = F_n(t) \quad t \in \mathbb{R}$$

$$\forall n \in \mathbb{Z}^+.$$

<u>exercise</u>: Think about 1.1.19 and 1.1.18 again. $\searrow$ [PT]

<u>exercise</u>: Read 1.1.14 $\longrightarrow$

(look up Caratheodory extension thm in [CI])