Math 137 10/6/08

<u>Thm:</u> $\{f_n\}_{n=1}^{\infty}$ $f_n \geq 0$ msble on $(E, \mathcal{B}, \mu)$

$f_n \nearrow f$ p.w $n \to \infty$

$$\Rightarrow \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

<u>Pf:</u>

$$\int f_n \, d\mu \leq \int f \, d\mu \qquad \forall n$$

so $\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu.$

<u>For the reverse inequality</u>:

For each m

Choose $\varphi_{m,n} \nearrow f_m$ $n \to \infty$.
non-negative, simple and msble.
and let

$\psi_n = \varphi_{1,n} \vee \dots \vee \varphi_{n,n}$ $n \geq 1$.

= world record up to time n.

then

$0 \leq \psi_n \leq \psi_{n+1}$ $\varphi_{m,n} \leq \psi_n \leq f_n$.

and ψ_n are simple. $\forall\, 1 \leq m \leq n$.

$\therefore \quad f_m \leq \lim_{n\to\infty} \varphi_n \leq f \qquad \forall m.$

and $\therefore \quad \lim_{n\to\infty} \varphi_n = f.$ so that.

$$\int f \, d\mu = \lim_{n\to\infty} \int \varphi_n \, d\mu.$$

but $\quad \varphi_n \leq f_n \quad$ so

$$\int \varphi_n \leq \int f_n \quad \text{and.}$$

$$\lim_{n\to\infty} \int \varphi_n \leq \lim_{n\to\infty} \int f_n$$

$$\therefore \int f \, d\mu \leq \lim_{n\to\infty} \int f_n.$$

Theorem: Fatou's lemma.

$\{f_n\}_{n=1}^{\infty}$ mble fcns on $(E, \mathcal{B}, \mu)$

$f_n \geq 0 \qquad n \geq 1.$

$$\Rightarrow \int \underline{\lim}_{n\to\infty} f_n \, d\mu \leq \underline{\lim}_{n\to\infty} \int f_n \, d\mu.$$

If $\exists$ g which is μ-integrable s.t. $f_n \leq g$ $\forall$ $n \geq 1$. then

$$\int \overline{\lim}_{n\to\infty} f_n \, d\mu \geq \overline{\lim}_{n\to\infty} \int f_n \, d\mu.$$

Pf. Assume $f_n \geq 0$

Put $h_m = \inf_{n \geq m} f_n$, so $f_m \geq h_m$ $\forall m$

and $h_m \nearrow \underline{\lim}_{n\to\infty} f_n$.

By M.C.T. $\int \underline{\lim}_{n\to} f_n d\mu = \lim_{m\to} \int h_m d\mu \leq \underline{\lim}_{m\to\infty} \int f_m d\mu$

if the f_n are not non-negative but $f_n \le g$ for some μ-int. g then

$$f_n' = g - f_n \ge 0.$$

and $\underline{\lim}_{n\to\infty} f_n' = g - \overline{\lim}_{n\to\infty} f_n$

$$\underline{\lim}_{n\to\infty} \int f_n' \, d\mu = \int g \, d\mu - \overline{\lim}_{n\to\infty} \int f_n \, d\mu.$$

So $\int g - \overline{\lim} f_n \, d\mu \le \int g \, d\mu - \overline{\lim}_{n\to\infty} \int f_n \, d\mu.$

and $\therefore$ $\overline{\lim}_{n\to\infty} \int f_n \, d\mu \le \int \overline{\lim} f_n \, d\mu.$ ∎

<u>Theorem:</u> (Dominated Convergence).

$\{f_n\}_{n=1}^{\infty}$ msble

$f_n \to f$ a.e (μ).

$\exists\, g$ μ integrable s.t.

$|f_n| \leq g$ (a.e. μ) $\forall n \geq 1$.

$\Rightarrow$ f integrable and

$$\lim_{n\to\infty} \int |f_n - f|\, d\mu = 0.$$

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu.$$

<u>Pf:</u> Let $\hat{E}$ be the set of $x \in E$ for which $f(x) = \lim_{n\to\infty} f_n(x)$.

and $\sup_{n \geq 1} |f_n(x)| \leq g(x)$

$$\hat{E} = \{ x : \; f(x) = \lim_{n\to\infty} f(x) \text{ exists and } \sup_{n \geq 1} |f_n(x)| \leq g(x) \}.$$

$\hat{E}$ is msble and $\mu(\hat{E}^c) = 0$.

W.l.o.g. assume lim exist and dom. occurs for all x.

$$f = \lim_{n\to\infty} f_n$$

$$|f_n| \leq g \qquad |f - f_n| \leq 2g \quad \forall n.$$

So

$$\overline{\lim_{n\to\infty}} \left| \int f \, d\mu - \int f_n \, d\mu \right|$$

$$\leq \overline{\lim_{n\to\infty}} \int |f - f_n| \, d\mu. \leq \int \overline{\lim_{n\to\infty}} |f - f_n| \, d\mu = 0.$$

∎

<u>Theorem</u>: (Lieb's version of Fatou's lemma).

$(E, \mathcal{B}, \mu)$

$\{f_n\}_{n=1}^{\infty} \cup \{f\} \subseteq L^1(\mu)$.

$f_n \to f$ a.e. μ.

Then

$$\lim_{n\to\infty} \left| \|f_n\|_{L^1(\mu)} - \|f\|_{L^1(\mu)} - \|f_n - f\|_{L^1(\mu)} \right|$$

$$= \lim_{n\to\infty} \int \left| |f_n| - |f| - |f_n - f| \right| d\mu = 0.$$

In particular, if $\|f_n\|_{L^1(\mu)} \to \|f\|_{L^1(\mu)} < \infty$

then $\|f - f_n\|_{L^1(\mu)} \to 0$.

Pf: $\left| \|f_n\|_{L^1(\mu)} - \|f\|_{L^1(\mu)} - \|f_n - f\|_{L^1(\mu)} \right|$

$= \left| \int |f_n| d\mu - \int |f| d\mu - \int |f_n - f| d\mu \right|$

$\leq \int \left| |f_n| - |f| - |f_n - f| \right| d\mu.$

and.

$\left| |f_n| - |f| - |f_n - f| \right| \to 0 \quad$ a.e. μ

and $\left| |f_n| - |f| - |f_n - f| \right|$

$\leq \left| |f_n| - |f_n - f| \right| + |f|$

$\leq 2|f|.$

So $\int \left| |f_n| - |f| - |f_n - f| \right| d\mu \to 0$

by the dom. conv. thm.

examples

i) $f_n(x) = \chi_{[n,n+1]}(x)$

$f_n(x) \to 0$ a.e. (Lebesgue).

$\int f_n(x)dx = 1$ $\int \lim_{n\to\infty} f_n(x) = 0$.

(also $\int \underline{\lim}\, f_n(x) < \underline{\lim} \int f_n(x) dx$.).

ii) $\{f_n\}_{n=1}^{\infty}$ on $[0,1]$.

for $m \geq 0$ $0 \leq l < 2^m$

$$f_{2^m+l} = \chi_{[2^{-m}l,\, 2^{-m}(l+1)]}.$$

$$\overline{\lim_{n\to\infty}} f_n(x) = 1$$

$$\int_{[0,1]} f_n(x)\,dx = 2^{-m} \qquad 2^m \leq n < 2^{m+1}$$

so $\int_{[0,1]} f_n(x)\,dx \to 0$ $n \to \infty$.

$\therefore$ $\|f_n\| \to 0$ doesn't imply that $f_n \to 0$ a.e.

Convergence in Measure

$\{f_n\}_{n=1}^{\infty}$ msble on $(E, \mathcal{B}, \mu)$

converges in μ-measure to f.

$$f_n \xrightarrow{\mu} f$$

iff $\cancel{\mu(A_n)}$ for each $\epsilon > 0$

$$\mu\{x : |f_n - f| \geq \epsilon\} \to 0 \quad n \to \infty.$$

Notice: if $\|f_n - f\|_{L^1(\mu)} \to 0$ then

$$\mu\{x : |f_n - f| \geq \epsilon\} \leq \frac{1}{\epsilon}\int |f_n - f|\, d\mu \to 0.$$

for any $\epsilon > 0$

by Markov's inequality.

Notice also that if

$f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$ then

$$\mu\{|f-g| > \epsilon\} \le \mu\{|f-f_n| \ge \tfrac{\epsilon}{2}\} + \mu\{|f_n - g| \ge \tfrac{\epsilon}{2}\}$$

$\to 0$ as $n \to \infty$.

so $$\mu(f \neq g) = \lim_{\epsilon \to 0} \mu(|f-g| \ge \epsilon)$$

$$= \lim_{n \to \infty} \mu(|f-g| \ge \tfrac{1}{n}) = 0.$$

$\therefore$ $f = g$ a.e.

<u>Thm</u>: (~~Cauchy Criteria for convergence in measure~~).

Let $\{f_n\}_{n=1}^{\infty}$ $\mathbb{R}$-valued msble.

on $(E, \mathcal{B}, \mu)$.

then $\exists$ f s.t.

$$(*) \lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f-f_n| \geq \epsilon \right) = 0 \quad \epsilon > 0.$$

iff

$$(**) \lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_n-f_m| \geq \epsilon \right) = 0 \quad \epsilon > 0.$$

Moreover

$(*) \Rightarrow \quad f_n \to f$ a.e. and $f_n \xrightarrow{\mu} f$

and if $\mu(E) < \infty$ then

$f_n \to f$ iff $(*)$ holds.

So that on finite measure spaces.

μ a.e. convergence $\Rightarrow$ convergence in μ measure.

Pf: Let $\Delta = \{x \in E : \lim_{n\to\infty} f_n(x)$ d.n.e. in $\mathbb{R}\}$

For $m \geq 1$, $\epsilon > 0$

let

$$\Delta_m(\epsilon) = \{x : \sup_{n \geq m} |f_n - f_m| \geq \epsilon\}$$

then

$$\Delta = \bigcup_{\ell=1}^{\infty} \bigcap_{m=1}^{\infty} \Delta_m(\tfrac{1}{\ell}).$$

Now,

$$(**) \Rightarrow \mu\left(\bigcap_{m=1}^{\infty} \Delta_m(\epsilon)\right) = 0 \quad \forall \epsilon > 0.$$

and we have

$$\mu(\Delta) \leq \sum_{\ell=1}^{\infty} \mu\left(\bigcap_{m=1}^{\infty} \Delta_m(\tfrac{1}{\ell})\right)$$

so $(**) \Rightarrow \mu(\Delta) = 0$

i.e. f_n converges a.e.

If, also, f is $\mathbb{R}$-valued msble and $f_n \to f$ a.e. then.

$$\sup_{n \geq m} |f_n - f| \leq \sup_{n \geq m} |f_n - f_m| + |f_m - f|$$

$$\leq 2 \sup_{n \geq m} |f_n - f_m|. \quad \text{(a.e.)}.$$

so (**) $\Rightarrow$ existence of f s.t. (*) holds.

(Moreover) If (*) holds for some f then $f_n \to f$ a.e. and $f_n \xrightarrow{\mu} f$ and we get (**) from.

$$\mu\left(\sup_{n \geq m} |f_n - f_m| > \epsilon\right)$$

$$\leq \mu\left(\sup_{n \geq m} |f_n - f| > \frac{\epsilon}{2}\right) + \mu\left(\sup_{n \geq m} |f - f_m| \geq \frac{\epsilon}{2}\right).$$

Finally, if $\mu(E) < \infty$ and $f_n \to f$ a.e.

then by our earlier monotonicity result for measures.

$$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_n - f| \geq \epsilon\right)$$

$$= \mu\left(\bigcap_{m=1}^{\infty} \left\{\sup_{n\geq m} |f_n - f| \geq \epsilon\right\}\right) = 0.$$

$\forall\, \epsilon > 0.$

so (*) holds.

and $\therefore$ $f_n \to f$ in measure.