Math 137 10/1/08

Lemma: $(E, \mathcal{B})$

$\{f_n\}_{n=1}^{\infty}$ msble.

$\sup_{n\geq 1} f_n$, $\inf_{n\geq 1} f_n$, $\overline{\lim_{n\to\infty}} f_n$, $\underline{\lim}_{n\to\infty} f_n$

are all msble.

Pf:

$$\{\sup_{n\geq 1} f_n > \alpha\} = \bigcup_{n\geq 1} \{f_n > \alpha\}$$

$$\{\inf_{n\geq 1} f_n > \alpha\} = \bigcap_{n\geq 1} \{f_n > \alpha\}.$$

$$\underline{\lim}_{n\to\infty} f_n = \lim_{m\to\infty} \inf_{n\geq m} f_n$$

but $g_m = \inf_{n\geq m} f_n$ is msble.
(by the above.

and $g_m \nearrow \underline{\lim}_{n\to\infty} f_n$

So

$$\{\underline{\lim}_{n\to\infty} f_n > \alpha\} = \bigcup_{m=1}^{\infty} \{g_m > \alpha\}.$$

(and similarly for $\overline{\lim_{n\to\infty}} f_n$ ▨

If f ~~and~~ and g are msble.

then $\{f=g\} = \left(\{f-g>0\} \cup \{f-g<0\}\right)^c$

so $\{f=g\}$ is a msble set.

and $\therefore$

$\{\underline{\lim}_{n\to\infty} f_n = \overline{\lim}_{n\to\infty} f_n\}$ is msble

when $\{f_n\}_{n=1}^{\infty}$ are

msble fcns.

(so is its complement).

$\therefore$ $f(x) = \lim_{n\to\infty} f_n$ when limit exists.

$= 0$ else.

is a msble fcn.

Theorem: Monotone Convergence Theorem.

$\{f_n\}_{n=1}^{\infty} \geq 0$, msble

on $(E, \mathcal{B}, \mu)$

$f_n \nearrow f$ point wise $n \to \infty$

$$\Rightarrow \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

Pf: $\int f_n \, d\mu \leq \int f \, d\mu \quad \forall n$ so

$$\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu.$$

For the other direction, choose (for each m) non-neg., simple, msble

$$\varphi_{m,n} \nearrow f_m \quad \text{as} \quad n \to \infty.$$

and let $\psi_n = \varphi_{1,n} \vee \cdots \vee \varphi_{n,n}$

Then

$$\psi_n \le \psi_{n+1} \qquad \forall n$$

$$\varphi_{m,n} \le \psi_n \le f_n \qquad \forall\ 1 \le m \le n.$$

$$\therefore \quad \lim_{n\to\infty} \psi_n \le \lim_{n\to\infty} f_n = f.$$

and $\lim_{n\to\infty} \varphi_{m,n} = f_m \le \lim_{n\to\infty} \psi_n.$

so

$$\varphi_m \le \lim_{n\to\infty} \psi_n \le f \qquad \forall m.$$

and

$$\therefore \quad \lim_{n\to\infty} \psi_n = f. \qquad (\psi_n \nearrow f)$$

and

$$\int f \, d\mu = \lim_{n\to\infty} \int \psi_n \, d\mu. \le \lim_{n\to\infty} \int f_n d\mu$$

$\blacksquare$

Exercises on Cantor Sets

1. Prove that any perfect set in $\mathbb{R}^n$ is uncountable

2. Construct a subset of $[0,1]$ in the same way as the Cantor set by removing a subinterval of relative length θ, $0<\theta<1$.
(see exercise 2.1.20 in [CI])
Show that the resulting set is perfect and has measure zero.

3. ... at the k^{th} stage, each interval removed has length $\delta 3^{-k}$
$0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$ and contains no intervals.

4. ... at the k^{th} stage a subinterval of relative length θ_k, $0<\theta_k<1$, is removed from the center of each remaining interval. Show that the remainder has measure zero iff $\sum \theta_k = +\infty$.

5.(*) Construct a msble subset E of $[0,1]$ such that for every subinterval I, both $E\cap I$ and $I-E$ have positive measure. (Use #3).