Math 137 9/29/08.

<u>Last time:</u>

$(E, \mathcal{B})$ a measure space.

$f: E \to \bar{\mathbb{R}}$ is msble iff

$\{f > \alpha\} \in \mathcal{B} \quad \forall \alpha \in \bar{\mathbb{R}}^*$

iff ... etc.

f_1, f_2 msble $\Rightarrow$ $f_1+f_2,\ f_1-f_2,\ f_1 f_2,\ f_1+c,\ cf_1$ all msble.

$\{\min(f_1, f_2) > \alpha\} = \{f_1 > \alpha\} \cap \{f_2 > \alpha\}$.

so $\min(f_1, f_2)$ is msble.

$\{\max(f_1, f_2) > \alpha\} = \{f_1 > \alpha\} \cup \{f_2 > \alpha\}$

so $\max(f_1, f_2)$ is msble.

$\therefore \quad f^+ \equiv \max(f, 0)$

$f^- = -\min(f, 0)$

and $f^+ + f^- = |f|$ are all msble if f is.

Def: A function $f: E \to \overline{\mathbb{R}}$ is simple if it has finite range.

e.g. $\Gamma \subseteq E$

$$\chi_\Gamma(x) = \begin{cases} 1 & x \in \Gamma \\ 0 & x \in \Gamma^c \end{cases}$$

χ_Γ is called the characteristic function of Γ (by analysts) and is called the indicator function of Γ by probabilists.

Lebesgue Integrals of Simple Functions

$(E, \mathcal{B}, \mu)$ is a measure space.

$f \geq 0$ is simple and measurable on E.

$$\int_{E} f \, d\mu \equiv \sum_{\alpha \in \mathrm{Range}(f)} \alpha \, \mu\{f=\alpha\}$$

<u>*Lemma:*</u> $(E, \mathcal{B}, \mu)$ a measure space

$f \geq 0$ simple and msble

$$f = \sum_{l=1}^{n} \beta_l \chi_{\Delta_l}$$

with $\{\beta_1, \dots, \beta_n\} \subseteq [0, \infty]$

$\{\Delta_1, \dots, \Delta_n\} \subseteq \mathcal{B}$.

Then $\int f \, d\mu = \sum_{l=1}^{n} \beta_l \, \mu(\Delta_l)$

<u>Pf:</u> Put $\{\alpha_1, \dots, \alpha_m\}$ = distinct els. of Range(f).

$\Gamma_k = \{f = \alpha_k\}$ $\quad k = 1, \dots, m.$

We want to prove: $\sum_{k=1}^{m} \alpha_k \, \mu(\Gamma_k) = \sum_{l=1}^{n} \beta_l \, \mu(\Delta_l)$.

We have $\Gamma_k \cap \Gamma_{k'} = \emptyset$ for $k \neq k'$ (and $\bigcup_{k=1}^{m} \Gamma_k = E$).

so,

$$\sum_{l=1}^{n} \beta_l \mu(\Delta_l) = \sum_{l=1}^{n} \beta_l \sum_{k=1}^{m} \mu(\Delta_l \cap \Gamma_k)$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} \beta_l \mu(\Delta_l \cap \Gamma_k)$$

and we see that it suffices to show that

$$\alpha_k \mu(\Gamma_k) = \sum_{l=1}^{n} \beta_l \mu(\Delta_l \cap \Gamma_k).$$

for $1 \leq k \leq m$.

Since

$$\sum_{l=1}^{n} \beta_l \chi_{\Delta_l \cap \Gamma_k} = \sum_{l=1}^{n} \beta_l \chi_{\Delta_l} \cdot \chi_{\Gamma_k}$$

$$= \left(\sum_{l=1}^{n} \beta_l \chi_{\Delta_l} \right) \cdot \chi_{\Gamma_k}$$

$$= f \chi_{\Gamma_k} = \alpha_k \chi_{\Gamma_k}.$$

We may as well assume from the start that

$$(*) \qquad f = \alpha \chi_\Gamma = \sum_{l=1}^{n} \beta_l \chi_{\Delta_l}$$

where $\Delta_l \subseteq \Gamma$ (and Δ_l msble)

for each $l = 1, \ldots, n$.

If $\alpha = 0$, then $\beta_l = 0$ unless $\Delta_l = \phi$

so the result is trivial in that case.

If $\alpha = +\infty$, then $\beta_l = +\infty$ for at least

one $l \in \{1, \ldots, n\}$ s.t $\Delta_l \neq \phi$ and

the result holds in that case as well.

Dividing both sides of $(*)$ by α,

we have reduced matters to proving

that for $\Gamma \in \mathcal{B}$ and $\{\Delta_1, \ldots \Delta_n\} \subseteq \mathcal{B}[\Gamma]$,

$$\sum_{l=1}^{n} \beta_l \chi_{\Delta_l} = \chi_\Gamma$$

$$\Rightarrow \quad \sum_{l=1}^{n} \beta_l \mu(\Delta_l) = \mu(\Gamma).$$

Let λ index the subsets of $\{1, \ldots, n\}$. (2^n possible values of λ).

For $x \in \Gamma$, let

$$\lambda(x) = \{ l \in \{1, \ldots, n\} : x \in \Delta_l \}.$$

Then $$\sum_{l \in \lambda(x)} \beta_l = 1 .$$

For fixed λ, let

$$\Delta_\lambda = \{ x \in \Gamma : \lambda(x) = \lambda \}$$

Then $\Delta_\lambda \cap \Delta_{\lambda'} = \emptyset$ if $\lambda \neq \lambda'$

$$\bigcup_\lambda \Delta_\lambda = \Gamma = \bigcup_{\{\lambda : \Delta_\lambda \neq \emptyset\}} \Delta_\lambda$$

and for each l

$$\Delta_l = \bigcup_{\{\lambda : l \in \lambda, \Delta_\lambda \neq \emptyset\}} \Delta_\lambda .$$

Also, if $\Delta_\lambda \neq \emptyset$, then $\sum_{l \in \lambda} \beta_l = 1$.

We have

$$\mu(\Delta_\ell) = \sum_{\{\lambda : \ell \in \lambda,\, \Delta_\lambda \neq \emptyset\}} \mu(\Delta_\lambda)$$

and therefore

$$\sum_{\ell=1}^{n} \beta_\ell \mu(\Delta_\ell) = \sum_{\ell=1}^{n} \beta_\ell \sum_{\{\lambda : \ell \in \lambda,\, \Delta_\lambda \neq \emptyset\}} \mu(\Delta_\lambda)$$

$$= \sum_{\{\lambda : \Delta_\lambda \neq \emptyset\}} \mu(\Delta_\lambda) \sum_{\ell \in \lambda} \beta_\ell$$

$$= \sum_{\{\lambda : \Delta_\lambda \neq \emptyset\}} \mu(\Delta_\lambda) = \mu(\Gamma).$$

$\blacksquare$

From the previous lemma, linearity of the integral for non-negative simple measurable functions follows easily...

Lemma: $f \geq 0$, $g \geq 0$ simple, msble on $(E, \mathcal{B}, \mu)$.

$$\alpha, \beta \in [0, \infty]$$

Then $\alpha f + \beta g \geq 0$ is simple, msble

and

$$\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu.$$

Pf: Let $\{\alpha_1, \ldots, \alpha_m\} = \text{Range}(f)$, distinctly.

$\{\beta_1 \ldots \beta_n\} = \text{Range}(g)$, " .

and write

$$\alpha f + \beta g = \sum_{k=1}^{m+n} \gamma_k \chi_{\Delta_k}$$

$$\gamma_k = \alpha \alpha_k \qquad 1 \leq k \leq m$$
$$\Delta_k = \{f = \alpha_k\} \qquad 1 \leq k \leq m$$
$$\gamma_k = \beta \beta_{k-m} \qquad m+1 \leq k \leq m+n$$
$$\Delta_k = \{g = \beta_{k-m}\} \qquad m+1 \leq k \leq m+n.$$

then

$$\int(\alpha f+\beta g)\,d\mu = \sum_{k=1}^{m+n} \gamma_k\, \mu(\Delta_k)$$

$$= \alpha \sum_{k=1}^{m} \alpha_k\, \mu\{f=\alpha_k\}$$

$$+ \beta \sum_{l=1}^{n} \beta_l\, \mu\{g=\beta_l\}.$$

$$= \alpha\int f\,d\mu + \beta\int g\,d\mu.$$

$\boxtimes$

<u>Corollary</u> if $0 \le f \le g$ (simple msble) ~~nasm~~ and if $\int f\,d\mu < \infty$

then $\int(g-f)\,d\mu + \int f\,d\mu = \int g\,d\mu$

so $\int(g-f)\,d\mu = \int g\,d\mu - \int f\,d\mu.$

$\therefore \quad \int f\,d\mu \le \int g\,d\mu.$

Notice also that if $\int f\,d\mu = +\infty$ then $\int g\,d\mu = +\infty$.

Extending the def'n of the L-integral to non-negative msble fcns.

$f \geq 0$, msble on $(E, \mathcal{B}, \mu)$.

Consider, for each $n \geq 1$,

$$\varphi_n = \sum_{k=0}^{4^n-1} 2^{-n} \cdot k \, \chi_{\{2^n f \in [k, k+1)\}} + 2^n \chi_{\{f \geq 2^n\}}.$$

For each $n \geq 1$,

$\varphi_n \geq 0$ is msble and simple

so we have defined its integral.

Also $\varphi_n \nearrow f$ (and uniformly on $(\overline{\mathbb{R}}, \rho)$).

and since $\varphi_n \leq \varphi_{n+1}$ we have

$$\int \varphi_n \, d\mu \leq \int \varphi_{n+1} \, d\mu$$

so that

$$\lim_{n\to\infty} \int \varphi_n \, d\mu \quad \text{exists.}$$

We want to <u>define</u>

$$\int f \, d\mu = \lim_{n\to\infty} \int \varphi_n \, d\mu$$

but we need to check that the definition is independent of the choice of the approximating sequence $\{\varphi_n\}$.

<u>Lemma:</u> $(E, \mathcal{B}, \mu)$ a measure space

$\{\varphi_n\}_{n=1}^{\infty}$, $\psi \ge 0$ mble, simple on $(E, \mathcal{B})$

$$\varphi_n \le \varphi_{n+1} \quad \forall n$$

$$\psi \le \lim_{n\to\infty} \varphi_n$$

Then, $\int \psi \, d\mu \le \lim_{n\to\infty} \int \varphi_n \, d\mu$.

Before proving the lemma, notice that it gives us what we want.

If $f \geq 0$ is msble on $(E, \mathcal{B})$

and $\{\psi_n\}_{n=1}^{\infty} \geq 0$ $\quad \psi_n \leq \psi_{n+1}$

are simple, msble on $(E, \mathcal{B})$

with $\psi_n \nearrow f$.

then $\lim_{n\to\infty} \int \psi_n \, d\mu = \lim_{n\to\infty} \int \varphi_n \, d\mu$

where the φ_n are as before.

This is because, for fixed m

we have

$$\varphi_m \leq \lim_{n\to\infty} \psi_n \quad \text{and} \quad \psi_m \leq \lim_{n\to\infty} \varphi_n$$

so by the lemma.

$$\int \varphi_m \, d\mu \leq \lim_{n\to\infty} \int \psi_n \, d\mu$$

and $\int \psi_m \, d\mu \leq \lim_{n\to\infty} \int \varphi_n \, d\mu$.

and letting $m \to \infty$ gives the result.

<u>Pf:</u> (of lemma).

<u>Case 1</u> $\underline{\mu(\{\psi = \infty\}) > 0}$.

For each $M < \infty$,

$$\mu(\varphi_n > M) \nearrow \mu\Big(\bigcup_{n=1}^{\infty}\{\varphi_n > M\}\Big)$$
$$\geq \mu(\psi > M)$$
$$\geq \mu(\psi = \infty) = \epsilon > 0.$$

(for some $\epsilon > 0$).

So $\lim_{n\to\infty} \int \varphi_n \, d\mu \geq \lim_{n\to\infty} M \mu(\varphi_n > M)$

$$\geq M\epsilon \qquad \forall M > 0$$

$\therefore \lim_{n\to\infty} \int \varphi_n \, d\mu = +\infty = \int \psi \, d\mu$.

<u>Case 2</u> $\underline{\mu(\psi > 0) = \infty}$

ψ is simple, so $\exists \epsilon > 0$ s.t.

$\psi > 0 \Rightarrow \psi > \epsilon$.

$\therefore$

$$\mu(\varphi_n > \epsilon) \nearrow \mu\left(\bigcup_{n=1}^{\infty}\{\varphi_n > \epsilon\}\right)$$

$$\geq \mu(\psi > 0) = \infty .$$

Since $\varphi_n \geq \epsilon \chi_{\{\varphi_n \geq \epsilon\}}$, we have

$$\lim_{n\to\infty} \int \varphi_n \, d\mu \geq \lim_{n\to\infty} \epsilon \mu(\varphi_n \geq \epsilon) = \infty$$

$$= \int \psi \, d\mu .$$

<u>Case 3</u> $\mu(\psi = \infty) = 0$ <u>and</u> $\mu(\psi > 0) < \infty$.

Let $\hat{E} = \{0 < \psi < \infty\}$.

Then $\mu(\hat{E}) < \infty$ and

$$\int \psi \, d\mu = \int_{\hat{E}} \psi \, d\mu .$$

Also, $\int \varphi_n \, d\mu \geq \int_{\hat{E}} \varphi_n \, d\mu \left(\equiv \int_E \chi_{\hat{E}} \varphi_n \, d\mu\right)$

$\forall n \geq 1$.

So we may assume that

$$E = \hat{E}.$$

Then $\mu(E) < +\infty$ and $\exists\ \epsilon > 0,\ M > 0$

s.t $E = \{ \epsilon \le \psi \le M \}$.

Let $0 < \delta < \epsilon$ and let $E_n = \{ \varphi_n \ge \psi - \delta \}$

Then $E_n \nearrow E$ and $\therefore$

$$\lim_{n\to\infty} \int \varphi_n \, d\mu \ \ge \ \lim_{n\to\infty} \int_{E_n} \varphi_n \, d\mu$$

$$\ge \lim_{n\to\infty} \Big[\int_{E_n} \psi \, d\mu - \delta \mu(E_n) \Big]$$

$$= \lim_{n\to\infty} \Big[\int_E \psi \, d\mu - \int_{E_n^c} \psi \, d\mu - \delta \mu(E_n) \Big]$$

$$\ge \lim_{n\to\infty} \Big[\int_E \psi \, d\mu - M \mu(E_n^c) - \delta \mu(E_n) \Big]$$

$$= \int_E \psi \, d\mu - M \lim_{n\to\infty} \mu(E_n^c) - \delta \mu(E).$$

$$= \int_E \psi \, d\mu - \delta \mu(E).$$

As $\delta > 0$ is arbitrary, we are done ∎

We define the Lebesgue integral of a msble $f \geq 0$ as suggested earlier and note the independence of the defn from the approximating sequence φ_n.

Linearity for msble $f \geq 0$, $g \geq 0$ is an easy consequence.

<u>Lemma:</u> $f \geq 0$, $g \geq 0$ msble on $(E, \mathcal{B}, \mu)$
$\alpha, \beta \in [0, \infty]$

Then,

$$\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu.$$

If $f \leq g$ and $\int f\, d\mu < +\infty$ then

$$\int (g - f)\, d\mu = \int g\, d\mu - \int f\, d\mu.$$

In any case, $f \leq g \Rightarrow \int f\, d\mu \leq \int g\, d\mu$.

The next theorem gives a quantative statement of how $\int f\,d\mu$ for msble $f \geq 0$ reflects the size of f.

<u>Thm</u>: (Markov's inequality, Chebyshev's inequality)

$f \geq 0$ msble on $(E, \mathcal{B}, \mu)$, $\lambda > 0$

Then

- $\mu(f \geq \lambda) \leq \frac{1}{\lambda} \int_{\{f \geq \lambda\}} f\,d\mu \leq \frac{1}{\lambda} \int f\,d\mu$.

- $\int f\,d\mu = 0 \Leftrightarrow \mu(f > 0) = 0$

- $\int f\,d\mu < \infty \Rightarrow \mu(f = \infty) = 0$.

<u>Pf</u>: $\lambda \chi_{\{f \geq \lambda\}} \leq \chi_{\{f \geq \lambda\}} f \leq f$

so $\lambda \mu\{f \geq \lambda\} \leq \int_{\{f \geq \lambda\}} f\,d\mu \leq \int_E f\,d\mu$.

Since $\mu(f \geq \epsilon) \nearrow \mu(f > 0)$ as $\epsilon \searrow 0$

$\int f \, d\mu = 0 \Rightarrow \mu(f > 0) = 0$. (by the Chebyshev inequality)

(and the converse is clear)

If $\int f \, d\mu = M < +\infty$ then

$$\mu(f \geq \lambda) \leq \frac{M}{\lambda} \qquad \forall \lambda > 0$$

so $\mu(f = \infty) \leq \lim_{\lambda \to \infty} \mu(f \geq \lambda) = 0$.

Def'n of Lebesgue Integral for almost arbitrary mble f on $(E, \mathcal{B}, \mu)$.

$\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are both defined. If one or both are finite we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

and say that $\int f \, d\mu$ exists.

remarks:

- If $\int f\,d\mu$ exists then so does

$$\int f\chi_\Gamma\,d\mu \equiv \int_\Gamma f\,d\mu \qquad \forall \Gamma \in \mathcal{B}$$

and

$$\int_{\Gamma_1\cup\Gamma_2} f\,d\mu = \int_{\Gamma_1} f\,d\mu + \int_{\Gamma_2} f\,d\mu$$

if $\Gamma_1, \Gamma_2 \in \mathcal{B}$ are disjoint.

- If $\int f\,d\mu$ exists, then

$$\left|\int f\,d\mu\right| = \left|\int f^+ d\mu - \int f^- d\mu\right| \le \left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right|$$
$$= \int |f|\,d\mu.$$

so (e.g)

$$\int_\Gamma f\,d\mu = 0 \quad \text{if } \mu(\Gamma) = 0.$$

- There is no definition made when

$$\int f^+ d\mu = \int f^- d\mu = \infty.$$

Lemma: (linearity)

f, g msble on $(E, \mathcal{B}, \mu)$

$\int f\,d\mu$, $\int g\,d\mu$ exist and

are either both finite, both $+\infty$, or both $-\infty$

i.e. $\left(\int f\,d\mu, \int g\,d\mu\right) \in \hat{\mathbb{R}}^2$.

Then $\mu\left(f \otimes g \notin \hat{\mathbb{R}}^2\right) = 0$

$$\int_{\{f \otimes g \in \hat{\mathbb{R}}^2\}} (\dot{f+g})\,d\mu$$ exists and

$$\int_{\{f \otimes g \in \hat{\mathbb{R}}^2\}} f+g\,d\mu = \int f\,d\mu + \int g\,d\mu$$

Pf: (left for reading)

In class we will mainly discuss the notation.

<u>Defn</u> for f msble on $(E, \mathcal{B}, \mu)$

$$\|f\|_{L^1(\mu)} = \int |f| \, d\mu.$$

$f : E \to \overline{\mathbb{R}}$ is <u>μ-integrable</u>

if f is msbl on $(E, \mathcal{B})$ and

$\|f\|_{L^1(\mu)} < +\infty.$

$L^1(\mu) = L^1(E, \mathcal{B}, \mu) = \{ \mathbb{R}\text{-valued integrable fcns} \}.$

if f is μ-integrable then

$\mu\{|f| = \infty\} = 0$. so

$\chi_{\{|f| < \infty\}} f \in L^1(\mu)$ and

$\|f - \chi_{\{|f|<\infty\}} f\|_{L^1(\mu)} = 0.$

so there is no harm in considering only $\mathbb{R}$ valued fcns in $L^1(\mu)$.

Lemma: $L^1(\mu)$ (in $(E, \mathcal{B}, \mu)$)

is a vector space and

$$\|\alpha f + \beta g\|_{L^1(\mu)} \le |\alpha| \|f\|_{L^1(\mu)} + |\beta| \|g\|_{L^1(\mu)}$$

$\forall \alpha, \beta \in \mathbb{R}$, $f, g \in L^1(\mu)$.

Pf: $|\alpha f + \beta g| \le |\alpha||f| + |\beta||g|$.

For any $f, g, h \in L^1(\mu)$ we have

$$\|f - h\|_{L^1(\mu)} \le \|f - g\|_{L^1(\mu)} + \|g - h\|_{L^1(\mu)}$$

and $\| \; \|_{L^1(\mu)}$ is almost a metric.

To actually make it a metric we need

$$\|f\|_{L^1(\mu)} = 0 \implies f = 0.$$

We set $f \overset{\mu}{\sim} g \qquad (f, g \in L^1(\mu))$

if $\mu(f - g \neq 0) = 0$.

$\| \ \|_{L^1(\mu)}$ becomes a metric on the set of equivalence classes.