Math 137 9/24/08.

Theorem: • The Lebesgue measurable sets in $\mathbb{R}^N$ form a σ-algebra.

• $\Gamma \subset \mathbb{R}^N$ is Lebesgue measurable iff $\exists\, A \in \mathcal{F}_\sigma$ and $B \in \mathcal{G}_\delta$ s.t

$$A \subseteq \Gamma \subseteq B \quad \text{and} \quad |B \setminus A| = 0.$$

$$(\therefore\ |A| = |B| = |\Gamma|).$$

Pf: The 1st assertion was proved last time.

If Γ is Lebesgue measurable we can choose $G_n \in \mathcal{G}$, $\Gamma \subset G_n$ and $|G_n \setminus \Gamma| < 1/n$. Then $B = \bigcap_n G_n$ has $\Gamma \subseteq B$ and $|B \setminus \Gamma| \leq |G_n \setminus \Gamma| \ \forall n$

so that $|B \setminus \Gamma| = 0$.

In the same way, choose $B' \in \mathcal{G}_\delta$ s.t.

$|B' \setminus \Gamma^c| = 0$ and $\Gamma^c \subseteq B'$.

Let $A = (B')^c$. Then $A \in \mathcal{F}_\sigma$, $A \subseteq \Gamma$ and

$|\Gamma \setminus A| = |A^c \setminus \Gamma^c| = 0$.

$B \setminus A = B \setminus \Gamma \cup \Gamma \setminus A$ so $|B \setminus A| = 0$.

Conversely, if $\exists\ A \in \mathcal{F}_\sigma$ and $B \in \mathcal{G}_\delta$ s.t

$$A \subseteq \Gamma \subseteq B \quad \text{and} \quad |B \setminus A| = 0$$

then $\Gamma = (\Gamma \setminus A) \cup A$

and $|\Gamma \setminus A|_e \leq |B \setminus A| = 0.$

so Γ is measurable.

Non-measurable sets

The axiom of choice:

If $\{X_\alpha\}_{\alpha \in A}$ is an indexed family of sets. There is a map

$$f : A \to \bigcup_{\alpha \in A} X_\alpha$$

such that $f(\alpha) \in X_\alpha$ for every $\alpha \in A$.

Note: The (non-empty) set of such mappings is denoted

$$\prod_{\alpha \in A} X_\alpha$$

and called the Cartesian Product of $\{X_\alpha\}_{\alpha \in A}$

In other words: There is a set consisting of exactly one element taken from each X_α.

<u>Lemma</u>: If $\Gamma \subset \mathbb{R}$ is measurable and

$|\Gamma| > 0$ then $\exists\, \delta > 0$ s.t

$$(-\delta, \delta) \subset \Gamma - \Gamma \equiv \{y - x : x, y \in \Gamma\}.$$

<u>Pf</u>: (postponed)

First let's see why the Lemma shows there are non-measurable sets.

Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if $y - x \in \mathbb{Q}$.

For each $x \in \mathbb{R}$, the equivalence class $[x]^\sim$ of x is $x + \mathbb{Q}$, so there are uncountably many equivalence classes.

Use the axiom of choice to create a set A which contains precisely one element from each of the equivalence classes $[x]^{\sim}$, $x \in \mathbb{R}$.

Then $A - A \subset (\mathbb{Q}^c \cup \{0\})$, so $A - A$ can not contain $(-\delta, \delta)$ for any $\delta > 0$. But we can't have $|A - A|_e = 0$ since

$$\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (q + A)$$

(every $x \in \mathbb{R}$ is in some equivalence class, $\therefore$ equivalent to some element of A).

By the lemma (to be proved), A can not be measurable.

Pf: (of the lemma)

w.m.a. $|\Gamma| < +\infty$.

Choose $G \in \mathcal{G}$ s.t $|G \setminus \Gamma| < \frac{1}{3}|\Gamma|$

and let $G = \bigcup_{i=1}^{\infty} I_i$ where the I_i

are non-overlapping closed intervals.

Then $\sum_{i=1}^{\infty} |I_i \cap \Gamma| = \sum_{i=1}^{\infty} |\mathring{I}_i \cap \Gamma| = |\Gamma|$

and $|G| = |\Gamma| + |G \setminus \Gamma| < 4/3\,|\Gamma|$

so that $|\Gamma| \geq \frac{3}{4}|G| = \frac{3}{4} \sum_{i=1}^{\infty} |I_i|$

$= \frac{3}{4} \sum_{i=1}^{\infty} |I_i^\circ|$.

$\therefore$

$\exists\ i$ s.t.

$$|I_i^\circ \cap \Gamma| \geq \frac{3}{4}|I_i^\circ|.$$

Let $A = \mathring{I}_i \cap \Gamma$ for ~~this~~ one such choice of i.

and let $I_i = [a,b]$.

We claim that if $|d|>0$ is sufficiently small, then we must have $(d+A)\cap A \neq \emptyset$. (so that $d \in A-A$).

With this in mind, suppose that

$$(d+A)\cap A = \emptyset \quad \text{for some } d \in \mathbb{R}.$$

Then

$$2|A| = |d+A| + |A| = |(d+A)\cup A|$$

$$\leq \left|(d+(a,b))\cup(a,b)\right|.$$

If $d>0$ then

$$\big(d+(a,b)\big)\cup(a,b) \subset (a,b+d)$$

and if $d<0$ then

$$\big(a+(a,b)\big)\cup(a,b) \subset (a+d,b)$$

so that in either case.

$$\left|(d+(a,b))\cup(a,b)\right| \leq (b-a)+|d|.$$

It follows that if $(d+A)\cap A = \emptyset$ then

$$\frac{6}{4}(b-a) \le 2|A| \le (b-a) + |d|$$

from which we have $|d| \ge \frac{b-a}{2}$.

$$\therefore \quad |d| < \frac{b-a}{2} \Rightarrow (d+A)\cap A \neq \emptyset$$

i.e. $A - A$ contains $\left(-\left(\frac{b-a}{2}\right), \left(\frac{b-a}{2}\right)\right)$ ∎

<u>Theorem</u>: Assuming the axiom of choice, every $\Gamma \subseteq \mathbb{R}$ with $|\Gamma|_e > 0$ contains a non-measurable subset.

<u>Pf</u>: Let A be the non-measurable set constructed earlier. Then

$$\Gamma = \bigcup_{q\in\mathbb{Q}} \Gamma \cap (q+A) \qquad \text{and}$$

$$0 < |\Gamma|_e \le \sum_{q\in\mathbb{Q}} |\Gamma\cap(q+A)|_e$$

There must be a $q \in \mathbb{Q}$ s.t.

$|\Gamma \cap (q+A)|_e > 0$, but

by the previous lemma,

$\Gamma \cap (q+A)$ can't be measurable

since its difference set contains

no interval $\blacksquare$

- Measurable functions, equivalence of defn's.
- measurability of

  $f+c$, cf, $f+g$, $f-g$, fg.