Math 137 9/22.

Lemma: • $\Gamma \subseteq \mathbb{R}^N$

i) $|\Gamma|_e = \inf \{ |G|_e : \Gamma \subseteq G \in \mathcal{G} \}$

ii) For each $\Gamma \subseteq \mathbb{R}^N$, $\exists\, B \in \mathcal{G}_\delta$ s.t

$\Gamma \subseteq B$ and $|\Gamma|_e = |B|_e$.

Pf:

For any G s.t $\Gamma \subseteq G$

$$|\Gamma|_e \leq |G|_e$$

so that $|\Gamma|_e \leq \inf \{ |G|_e : \Gamma \subseteq G \in \mathcal{G} \}$.

For the reverse inequality, we show that given $\epsilon > 0$, there is $G \in \mathcal{G}$ with $\Gamma \subseteq \mathcal{G}$ and $|G|_e \leq |\Gamma|_e + \epsilon$.

From this, it follows that

$$\inf \{ |G|_e : \Gamma \subseteq G \in \mathcal{G} \} \leq |\Gamma|_e + \epsilon$$

and we will have what we need since $\epsilon > 0$ is arbitrary.

If $|\Gamma|_e = +\infty$, the equality is trivial, so assume that $|\Gamma|_e < +\infty$. With $\epsilon > 0$ given, choose a covering $\mathcal{C}$ s.t.

$$\sum_{I \in \mathcal{C}} |I| \le |\Gamma|_e + \epsilon/2 .$$

Let $\mathcal{C}$ be written as $\{I_\ell\}_{\ell=1}^{\infty}$ and choose rectangles I_ℓ' s.t

$$\mathring{I}_\ell' \supseteq I_\ell \quad \text{and} \quad |I_\ell'|_e \le |I_\ell|_e + 2^{-\ell}\left(\tfrac{\epsilon}{2}\right).$$

Let $G = \bigcup_{\ell=1}^{\infty} \mathring{I}_\ell'$. Then G is open, $\Gamma \subseteq G$ and $|G|_e \le \sum_{\ell=1}^{\infty} |\mathring{I}_\ell'|_e \le \sum_{\ell=1}^{\infty} |I_\ell'|_e$

$$\le \sum_{\ell=1}^{\infty} |I_\ell|_e + 2^{-\ell}\left(\tfrac{\epsilon}{2}\right) \le |\Gamma|_e + \frac{\epsilon}{2} + \frac{\epsilon}{2} .$$

For ii), choose for each n, $G_n \in \mathcal{G}$ s.t $|G_n|_e < |\Gamma|_e + \frac{1}{n}$ and $\Gamma \subseteq G_n$, then $B = \bigcap_n G_n$ has $\Gamma \subseteq B$ and $|B|_e \le |\Gamma|_e$ $\blacksquare$

<u>Def</u>: $\Gamma \subset \mathbb{R}^n$ is <u>Lebesgue</u> <u>Measurable</u> iff for each $\epsilon > 0$ $\exists\, G \in \mathcal{G}$ s.t $\Gamma \subseteq G$ and $|G \setminus \Gamma|_e < \epsilon$.

In this case, we write

$|\Gamma|$ for $|\Gamma|_e$. and refer to the measure, as Lebesgue Measure of Γ.

<u>Caution</u>: In general we may have

$$d(G \setminus \Gamma, \Gamma) = 0$$

and we can't conclude

$$|G|_e = |G \setminus \Gamma|_e + |\Gamma|_e .$$

We can only say

$$|G|_e \leq |G \setminus \Gamma|_e + |\Gamma|_e .$$

~~Al~~ So, while the previous lemma says we can choose G s.t $|G|_e \approx |\Gamma|_e$ and $\Gamma \subseteq G$.

it doesn't say that we can make $|G \setminus \Gamma|_e$ small.

In fact, as we will see, there are non-measurable sets in $\mathbb{R}^N$.

<u>Remarks</u>: • By definition, every open $G \subset \mathbb{R}^N$ is measurable.

• Also, if $|\Gamma|_e = 0$ then given $\epsilon > 0$ $\exists$ $G \in \mathcal{G}$ s.t $\Gamma \subseteq G$ and $|G|_e < \epsilon$.

Then $|G \setminus \Gamma|_e \leq |G|_e < \epsilon$

so Γ is measurable.

• If Γ is measurable, $\exists$ $B \in G_\delta$ s.t $\Gamma \subseteq B$ and $|B \setminus \Gamma|_e = 0$.

Just choose $G_n \in \mathcal{G}$ s.t $\Gamma \subseteq G_n$, and $|G_n \setminus \Gamma|_e < \frac{1}{n}$. Put $B = \bigcap_n G_n$

then $B \setminus \Gamma \subseteq G_n \setminus \Gamma$ for each n and $\Gamma \subseteq B$.

Lemma: $\{\Gamma_n\}_{n=1}^{\infty}$ measurable in $\mathbb{R}^N$.

- $\Gamma = \bigcup_1^{\infty} \Gamma_n$ is measurable and $|\Gamma| \le \sum_1^{\infty} |\Gamma_n|$

- Every rectangle in $\mathbb{R}^N$ is measurable.

Pf: Let $\epsilon > 0$ be given. For each n choose $G_n \in \mathcal{G}$ s.t.

$$\Gamma_n \subseteq G_n \quad \text{and} \quad |G_n \setminus \Gamma_n|_e < 2^{-n}\epsilon$$

$G = \bigcup_{n=1}^{\infty} G_n$ is open and

$$G \setminus \Gamma \subseteq \bigcup_1^{\infty} (G_n \setminus \Gamma_n) \quad \text{so that}$$

$$|G \setminus \Gamma|_e \le \sum_{n=1}^{\infty} |G_n \setminus \Gamma_n|_e < \epsilon \quad \blacksquare$$

<u>Lemma:</u> Γ measurable $\Rightarrow$ Γ^c measurable.

<u>Pf:</u> We first reduce to showing that

<u>Claim:</u> any closed subset of $\mathbb{R}^N$ is measurable.

The previous lemma and the claim imply that any $F \in \mathcal{F}_0$ is measurable.

If Γ is measurable, $\exists\, B \in \mathcal{G}_\delta$ s.t.

$$\Gamma \subseteq B \quad \text{and} \quad |B \setminus \Gamma|_e = 0.$$

Since $B^c \in \mathcal{F}_\sigma$ and

$$\Gamma^c = B^c \cup (B \setminus \Gamma),$$

the claim implies Γ^c is measurable.

<u>Pf:</u> (of claim).

If F is closed then $F = \bigcup_{n=1}^{\infty} (F \cap \overline{B(0,n)})$

where $B(0,n) = \{\vec{x} \in \mathbb{R}^N : |\vec{x}| < n\}$.

By the previous lemma, it is therefore enough to show that any compact $K \subset \mathbb{R}^N$ is mble.

Let $K \subset \mathbb{R}^n$ be compact and let $\epsilon > 0$ be given.

Choose an open set G s.t.

$K \subseteq G$ and

$$|G|_e < |K|_e + \epsilon.$$

$H = G \setminus K$ is open and $\therefore$

$H = \bigcup_{n=1}^{\infty} Q_n$ where the Q_n are non-overlapping closed dyadic cubes.

We have $\left| \bigcup_{n=1}^{m} Q_n \right| = \sum_{n=1}^{m} |Q_n|$ for each m.

And for each m, K and $\bigcup_{n=1}^{m} Q_n$ are disjoint compact sets and therefore are a positive distance apart.

It follows that

$$\left| K \cup \bigcup_{n=1}^{m} Q_n \right|_e = |K|_e + \left|\bigcup_{n=1}^{m} Q_n\right|_e.$$

and therefore

$$|G| \geq \left| \left(\bigcup_{n=1}^{m} Q_n \right) \cup K \right|_e = |K|_e + \sum_{n=1}^{m} |Q_n| .$$

Now,

$$\sum_{n=1}^{m} |Q_n| \leq |G| - |K|_e < \epsilon \quad \text{for each } m$$

so $$|H|_e \leq \sum_{n=1}^{\infty} |Q_n| < \epsilon .$$

$\therefore$ K is measurable.

<u>Lemma:</u> If $\{\Gamma_n\}_{n=1}^{\infty}$ are measurable in $\mathbb{R}^N$ and $\Gamma_m \cap \Gamma_n = \phi$ for $m \neq n$

then

$$\left| \bigcup_{1}^{\infty} \Gamma_n \right| = \sum_{1}^{\infty} |\Gamma_n|$$

<u>Pf:</u> Assume at first that the sets Γ_n are each bounded.

Given $\epsilon > 0$, choose $G_n \in \mathcal{G}$ s.t

$\Gamma_n^c \subseteq G_n$ and $|G_n \setminus \Gamma_n^c| < \frac{\epsilon}{2^n}$

Then $\quad K_n \equiv G_n^c \subseteq \Gamma_n$

and K_n is compact. The K_n have

$K_n \cap K_m = \phi$ if $n \neq m$ $\quad \therefore$

$$\operatorname{dist}(K_n, K_m) > 0 \quad n \neq m.$$

So $\quad \left| \bigcup_{m=1}^{n} K_m \right| = \sum_{m=1}^{n} |K_m| \quad$ for each n.

Also, $\quad \Gamma_n \setminus K_n = G_n \setminus \Gamma_n^c \quad$ so

$$|\Gamma_n \setminus K_n| < \epsilon/2^n$$

so that $\quad |\Gamma_n| < |K_n| + \epsilon/2^n. \quad$ for each n.

Now,

$$\sum_{m=1}^{\infty} |\Gamma_m| < \sum_{m=1}^{\infty}\left(|K_m| + \frac{\epsilon}{2^m}\right) = \left(\sum_{m=1}^{\infty} |K_m|\right) + \epsilon$$

$$= \lim_{n\to\infty} \left| \bigcup_{m=1}^{n} K_m \right| + \epsilon$$

$$\leq \left| \bigcup_{m=1}^{\infty} \Gamma_m \right| + \epsilon.$$

So $\sum_{m=1}^{\infty} |\Gamma_m| \leq \left| \bigcup_{m=1}^{\infty} \Gamma_m \right|$

and the reverse inequality always holds.

To remove the assumption that the $\{\Gamma_m\}$ are bounded let

$$R_1 = B(0,1) \quad , \ \ldots \quad R_{n+1} = B(0,n+1) \setminus B(0,n)$$

The sets R_j are disjoint, bounded and there union is $\mathbb{R}^N$.

We have,

$$(\Gamma_m \cap A_n) \cap (\Gamma_{m'} \cap A_{n'}) = \emptyset$$

$$\text{if } (m,n) \neq (m',n').$$

So,
$$\sum_{m=1}^{\infty} |\Gamma_m| = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\Gamma_m \cap A_n| = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\Gamma_m \cap A_n|$$

$$= \sum_{n=1}^{\infty} \left| \bigcup_{m=1}^{\infty} \Gamma_m \cap A_n \right| = \sum_{n=1}^{\infty} \left| \left(\bigcup_{m=1}^{\infty} \Gamma_m\right) \cap A_n \right|$$

$$= \left| \bigcup_{n=1}^{\infty} \left(\left(\bigcup_{m=1}^{\infty} \Gamma_m\right) \cap A_n \right) \right| = \left| \bigcup_{m=1}^{\infty} \Gamma_m \right| .$$

We have (mostly) proved the following

<u>Theorem:</u> The Lebesgue measurable sets in $\mathbb{R}^N$ form a σ-algebra.

$\Gamma \subset \mathbb{R}^N$ is Lebesgue measurable iff $\exists\ A \in \mathcal{F}_\sigma$ and $B \in \mathcal{G}_\delta$ s.t.

$$A \subseteq \Gamma \subseteq B \quad \text{and} \quad |B \setminus A| = 0,$$

in which case $|\Gamma| = |A| = |B|$.

<u>Pf:</u> We have shown the 1st assertion in the previous lemmas.

If $\exists\ A \in \mathcal{F}_\sigma$ and $B \in \mathcal{G}_\delta$ as above then

$$\Gamma = A \cup \Gamma \setminus A \quad \text{is measurable}$$

since $\Gamma \setminus A \subset B \setminus A$ and $|B \setminus A|_e = 0$.

If Γ is measurable choose $B \in G_\delta$ s.t.

$$\Gamma \subseteq B \quad \text{and} \quad |B \setminus \Gamma| = 0$$

Choose $A^c \in \mathcal{G}_\delta$ s.t $\Gamma^c \subset A^c$ and $|A^c \setminus \Gamma^c|_e = 0$

then $A \in F_\sigma$ and $|\Gamma \setminus A| = 0$. and $A \subseteq \Gamma$.

$B \setminus A = B \setminus \Gamma \cup \Gamma \setminus A$ so $|B \setminus A| = 0$.