Math 137 9/17/08.

<u>A technical point about Rectangles</u>

If $J = \prod_{k=1}^{N} [a_k, b_k]$

and $a_k = c_{k,0} \le \cdots \le c_{k,n_k} = b_k$

and S is the collection of rectangles of the form

$$R(m_1, \ldots, m_N) = \prod_{k=1}^{N} [c_{k,m_k-1}, c_{k,m_k}]$$

with $1 \le m_k \le n_k$, then

$$vol(J) = \sum_{R \in S} vol(R).$$

<u>Pf:</u> by induction on $q = \sum_{k=1}^{N} n_k$.

If $q = 1$ we are inserting c, $a_{k'} < c < b_{k'}$ for some k' and forming

$$J^+ = \{\vec{x} \in J : a_k' \le x \le c\}$$
$$J^- = \{\vec{x} \in J : c \le x \le b_k'\}.$$

The $J^{\pm}$ are rectangles and we have

$$\prod_{\substack{n=1\\ n\neq n'}}^{N} (b_n - a_n)\ (b_{n'} - a_{n'})$$

$$= \prod_{\substack{n=1\\ n\neq n'}}^{N} (b_n - a_n)\ \left((b_{n'} - c) + (c - a_{n'})\right)$$

so that $\quad vol(J) = vol(J^+) + vol(J^-)$.

For $q > 1$ we choose a single

$c_{n', m_{n'}} = c \quad$ to form J^+ and J^-

and use the inductive assumption on

J^+ and J^- separately.

<u>Lemma:</u> (next page)

Lemma:

- If $\mathcal{C}$ is a non-overlapping, finite collection of rectangles each of which is contained in a rectangle J

  then $$\mathrm{vol}(J) \geq \sum_{I \in \mathcal{C}} \mathrm{vol}(I).$$

- If $\mathcal{C}$ is any finite collection of rectangles and J is a rectangle for which

  $$J \subset \bigcup_{I \in \mathcal{C}} I$$

  then $$\mathrm{vol}(J) \leq \sum_{I \in \mathcal{C}} \mathrm{vol}(I)$$

Note:

if $I = (a_1, b_1) \times \dots \times (a_N, b_N)$

then $$\mathrm{vol}(I) \equiv \prod_{k=1}^{N} (b_k - a_k)$$

Pf: We may assume $I \subset J$ for each $I \in \mathcal{C}$ by replacing I with $I \cap J$.

Pf: cont

Write the rectangles $I \in \mathcal{C}$ as

$$I_j = \prod_{k=1}^{N} [a_k^{(j)}, b_k^{(j)}] \qquad j = 1, 2, \dots, M$$

and $$J = \prod_{k=1}^{N} [a_k^{(0)}, b_k^{(0)}]$$

For each k consider the collection of numbers

$$a_k^0, a_k^{(1)}, \dots a_k^{(M)}, b_k^1 \dots b_k^{(M)}, b_k^{(0)}.$$

~~and~~ Arrange them in increasing order. and Relabelling them to get

$$a_k^{(0)} = c_{k,0} \le c_{k,1} \le \dots \le c_{k,2M} \le c_{k(2M+1)} = b_k^{(0)}.$$

There are $(2M+1)^N$ rectangles of the form $R(m_1, \dots, m_N) \equiv \prod_{k=1}^{N} [c_{k,(m_k-1)}, c_{k,m_k}]$

with $1 \le m_k \le n_k$.

Denote the collect of such rectangles by $\mathcal{S}$. Then we have (as shown before) $\mathrm{vol}(J) = \sum_{R \in S} \mathrm{vol}(R)$.

For a given $I \in \mathcal{C}$, let $\mathcal{S}(I)$ be the collection of $R \in \mathcal{S}$ with $R \subseteq I$.

Then, in the same way, we have

$$\mathrm{vol}(I) = \sum_{R \in \mathcal{S}(I)} \mathrm{vol}(R).$$

$$\therefore \quad \sum_{I \in \mathcal{C}} \mathrm{vol}(I) = \sum_{I \in \mathcal{C}} \sum_{R \in \mathcal{S}(I)} \mathrm{vol}(R)$$

and we would have

$$\leq \sum_{R \in \mathcal{S}} \mathrm{vol}(R) = \mathrm{vol}(J)$$

if we knew that each $R \in \mathcal{S}$ was contained in at most one $\mathcal{S}(I)$ $\cancel{\in \mathcal{U}}$. for $I \in \mathcal{C}$.

This is not the case in general, but if $R \in \mathcal{S}$ is in $\mathcal{S}(I)$ $\cancel{\in \mathcal{U}}$ and also in $\mathcal{S}(I')$ $\cancel{\in \mathcal{U}}$

~~that since~~ (and the rectangles in $\mathcal{C}$ are non-overlapping)

where $I \neq I'$ are rectangles in $\mathcal{C}$.

then (with $R = R(m_1, \ldots, m_N)$) we must have $c_{k, m_k - 1} = c_{k, m_k}$ for at least one k. So $\mathrm{vol}(R) = 0$ and the inequality still holds.

If the collection $\mathcal{C}$ covers J (but may be overlapping)

then every $R \in \mathcal{S}$ is in $\mathcal{S}(I)$ for at least one $\cancel{I} \in \mathcal{C}$.

$\therefore$

$$\operatorname{vol}(J) = \sum_{R \in \mathcal{S}} \operatorname{vol}(R) \leq \sum_{I \in \mathcal{C}} \sum_{R \in \mathcal{S}(I)} \operatorname{vol}(R)$$

$$= \sum_{I \in \mathcal{C}} \operatorname{vol}(I)$$

∎

__Def'N__ Let $\Gamma \subset \mathbb{R}^N$.

$$|\Gamma|_e \equiv \inf \left\{ \sum_{I \in \mathcal{C}} \mathrm{vol}(I) : \Gamma \subseteq \bigcup_{I \in \mathcal{C}} \mathcal{C} \right\}$$

where the infimum is over all collections of rectangles $\mathcal{C}$ with $\Gamma \subseteq \bigcup_{I \in \mathcal{C}} \mathcal{C}$.

__Claim:__ $\overline{\mathcal{B}_{\mathbb{R}^N}}$ has the property that $|\cdot|_e$ is countably additive when restricted to it.

(lemmas are required to prove the claim)

__Lemma:__

If $\Gamma = \bigcup_{1}^{n} J_m$ where the J_m's are non-overlapping rectangles then $|\Gamma|_e = \sum_{m=1}^{n} \mathrm{vol}(J_m)$.

Proof: $|\Gamma|_e \leq \sum_1^n \mathrm{vol}(J_m)$ since

$\{J_m\}$ is an admissible covering.

For the reverse inequality, let $\epsilon > 0$

and consider any cover $\mathcal{C} = \{I_\ell\}_1^\infty$

of J by rectangles.

Let $\mathring{I}_\ell' \supseteq I_\ell$

with $\mathrm{vol}(I_\ell') \leq \mathrm{vol}(I_\ell) + 2^{-\ell}\epsilon$

be a slightly fattened up version of I_ℓ.

Since Γ is compact, it is covered

by finitely many of the I_ℓ'

and we may choose L s.t.

$\{I_1', \dots I_L'\}$ covers Γ.

Now we have

$$\sum_{m=1}^n \mathrm{vol}(J_m) \leq \sum_{m=1}^n \sum_{\ell=1}^L \mathrm{vol}(J_m \cap I_\ell').$$

$$\leq \sum_{\ell=1}^{L} vol(I_\ell') \leq \sum_{\ell=1}^{\infty} vol(I_\ell') \leq \sum_{\ell=1}^{\infty} (vol(I_\ell) + 2^{-\ell}\epsilon)$$

$$\leq \sum_{I \in \mathcal{C}} vol(I) + \epsilon.$$

As $\epsilon > 0$ was arbitrary we must have

$$\sum_{m=1}^{n} vol(J_m) \leq \sum_{I \in \mathcal{C}} vol(I)$$

and since $\mathcal{C}$ was an arbitrary covering we have.

$$\sum_{m=1}^{n} vol(J_m) \leq |\Gamma|_e.$$

Lemma (2.1.2)

If $\Gamma_1 \subseteq \Gamma_2$ then $|\Gamma_1|_e \leq |\Gamma_2|_e$.

If $\Gamma \subseteq \bigcup_1^\infty \Gamma_n$ then $|\Gamma|_e \leq \sum_1^\infty |\Gamma_n|_e$

If $\Gamma \subseteq \bigcup_1^\infty \Gamma_n$ and $|\Gamma_n|_e = 0 \quad \forall n \geq 1$ then

$|\Gamma|_e = 0$, (so $|\partial I|_e = 0$ if I is a rectangle)

If $\Gamma_1, \Gamma_2 \subset \mathbb{R}^N$ with $\inf\{|x-y| : x \in \Gamma_1, y \in \Gamma_2\} > 0$
then $|\Gamma_1 \cup \Gamma_2|_e = |\Gamma_1|_e + |\Gamma_2|_e$.

Pf: • Since any cover of Γ_2 is a cover of Γ_1 we have $|\Gamma_1|_e \le |\Gamma_2|_e$.

• If $\Gamma \subseteq \bigcup_1^\infty \Gamma_n$, let $\epsilon > 0$ be given and choose coverings $\mathcal{C}_n$ of each Γ_n

s.t $\displaystyle\sum_{I \in \mathcal{C}_n} |I|_e \le |\Gamma_n|_e + \epsilon \cdot 2^{-n}$

Then the union of all the $\mathcal{C}_n$ covers Γ so that

$$|\Gamma|_e \le \sum_{n=1}^\infty \sum_{I \in \mathcal{C}_n} |I_e| \le \sum_{n=1}^\infty \left(|\Gamma_n|_e + \epsilon \cdot 2^{-n}\right)$$

$$\le \sum_{n=1}^\infty |\Gamma_n|_e + \epsilon$$

Since $\epsilon > 0$ is arbitrary

$$|\Gamma|_e \le \sum_{n=1}^\infty |\Gamma_n|_e .$$

• If $\Gamma \subseteq \bigcup_1^\infty \Gamma_n$ and $|\Gamma_n|_e = 0$

then by the previous

$$|\Gamma|_e \le \sum_{n=1}^\infty |\Gamma_n|_e = 0.$$

If I is a rectangle then
∂I is a union of $2N$ rectangles with zero volume.
So $|\partial I|_e = 0$.

If $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^N$ with $\operatorname{dist}(\Gamma_1, \Gamma_2) = \delta > 0$
let $\mathcal{C}$ be any countable covering of $\Gamma_1 \cup \Gamma_2$ by rectangles. By subdividing rectangles in $\mathcal{C}$ we may assume that all $I \in \mathcal{C}$ have $\operatorname{diam}(I) < \frac{\delta}{10}$.

Let $\mathcal{C}_i = \{ I \in \mathcal{C} ; I \cap \Gamma_i \neq \emptyset \}$.

Then $\mathcal{C}_i$ covers Γ_i
and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

$\therefore$

$$|\Gamma_1|_e + |\Gamma_2|_e \le \sum_{I \in \mathcal{C}_1} |I|_e + \sum_{I \in \mathcal{C}_2} |I|_e \le \sum_{I \in \mathcal{C}} |I|_e.$$

Since the covering $\mathcal{C}$ was arbitrary, $|\Gamma_1|_e + |\Gamma_2|_e \le |\Gamma_1 \cup \Gamma_2|_e$.

The reverse inequality is clear, so we are done.

<u>Notation</u> $\mathcal{G} = \{\text{open sets}\}$

$\mathcal{F} = \{\text{closed sets}\}$

$\mathcal{G}_\delta = \{\text{countable intersections of open sets}\}$

$\mathcal{F}_\sigma = \{\text{countable unions of closed sets}\}$.

$\mathcal{G}_{\delta\sigma}$, $\mathcal{F}_{\sigma\delta}$ etc ...

<u>Remark</u>: • $\mathcal{F} \subset \mathcal{G}_\delta$

• $A \in \mathcal{G}_\delta \Rightarrow A^c \in \mathcal{F}_\sigma$

<u>Lemma</u>: $\Gamma \subset \mathbb{R}^N$

• $|\Gamma|_e = \inf \{ |G|_e : \Gamma \subseteq G \in \mathcal{G} \}$

• For each $\Gamma \subset \mathbb{R}^N$ $\exists$ $B \in \mathcal{G}_\delta$ s.t. $\Gamma \subseteq B$ and $|\Gamma|_e = |B|_e$.

Pf: for any G s.t $\Gamma \subseteq G$ we have $|\Gamma|_e \leq |G|_e$.

so $|\Gamma|_e \leq \inf \{ |G|_e : \Gamma \subseteq G \in \mathcal{G} \}$.

For the opposite inequality we may assume that $|\Gamma|_e < +\infty$.

Let $\epsilon > 0$ and choose a covering $\mathcal{E}$, of Γ s.t $\sum_{I \in \mathcal{E}} |I|_e \leq |\Gamma|_e + \epsilon/2$.

With $\mathcal{E} = \{ I_\ell \}_{\ell=1}^{\infty}$, choose I_ℓ' s.t.

$\mathring{I}_\ell' \supseteq I_\ell$ and $|I_\ell'|_e \leq |I_\ell|_e + 2^{-n}(\frac{\epsilon}{2})$

then $G = \bigcup_{\ell=1}^{\infty} \mathring{I}_\ell'$ is open, contains Γ

and has $|G|_e \leq \sum_{\ell=1}^{\infty} |I_\ell'|_e \leq |\Gamma|_e + \epsilon$.

For each n, we can now pick $G_n \in \mathcal{G}$ s.t. $\Gamma \subseteq G_n$ and

$|G_n|_e \leq |\Gamma|_e + 1/n$

Then $\quad B = \bigcap_{n=1}^{\infty} G_n \quad$ has

$$\Gamma \subseteq B \qquad \text{and} \qquad |B|_e \leq |\Gamma|_e$$

$$\therefore \quad |B|_e = |\Gamma|_e .$$