9/15/08 Math 137.

Let's collect some results and goals to remind us of where we are and what our plan is.

<u>Riemann integration</u>

Let $[a,b] \subset \mathbb{R}$ and $E \subset [a,b]$.

We say that E has Lebesgue Measure zero iff given $\epsilon > 0$ $\exists$ a collection of intervals ~~$\{I_n\}_{n=1}^{\infty}$ s.t. $E \subset \bigcup_{n=1}^{\infty} I_n$~~

~~and~~ $I_n = [a_n, b_n]$ $n = 1, 2, \ldots$

s.t. $E \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty}(b_n - a_n) < \epsilon$.

<u>Thm</u>: A bounded function f defined on $[a,b]$ is Riemann integrable on $[a,b]$ iff $\{x \in [a,b] : f \text{ is discontinuous at } x\}$ has Lebesgue Measure zero.

(and this is a rather restricted class of functions)

Riemann - Stieltjes Integrals

If $F: \mathbb{R} \to \mathbb{R}$ is any increasing function ($x < y \Rightarrow F(x) \leq F(y)$), which is continuous on the right, there is a unique measure μ_F on $\mathcal{B}_{\mathbb{R}}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant.

Conversely, if μ is a measure on $\mathcal{B}_{\mathbb{R}}$ which is finite on all bdd Borel sets and we define

$$F(x) = \mu((0,x]) \text{ if } x > 0$$

$$F(0) = 0$$

$$F(x) = -\mu((x,0]) \text{ if } x < 0.$$

then F is increasing, right continuous and $\mu = \mu_F$.

If μ is a Probability measure on $\mathcal{B}_{\mathbb{R}}$ we can take

$\mu = \mu_F$ where $F(x) = \mu((-\infty, x])$

and F is called the c.d.f

(cumulative distribution function) of μ.

Note that any R.V X defined on a Probability space $(\Omega, P, \mathcal{F})$ defines such a μ by $\mu(E) = P\{X \in E\}$ $\forall E \in \mathcal{B}_{\mathbb{R}}$.

Taking the completion μ_F of such a measure (as in our exercise) gives a so-called "Lebesgue-Stieltjes measure."

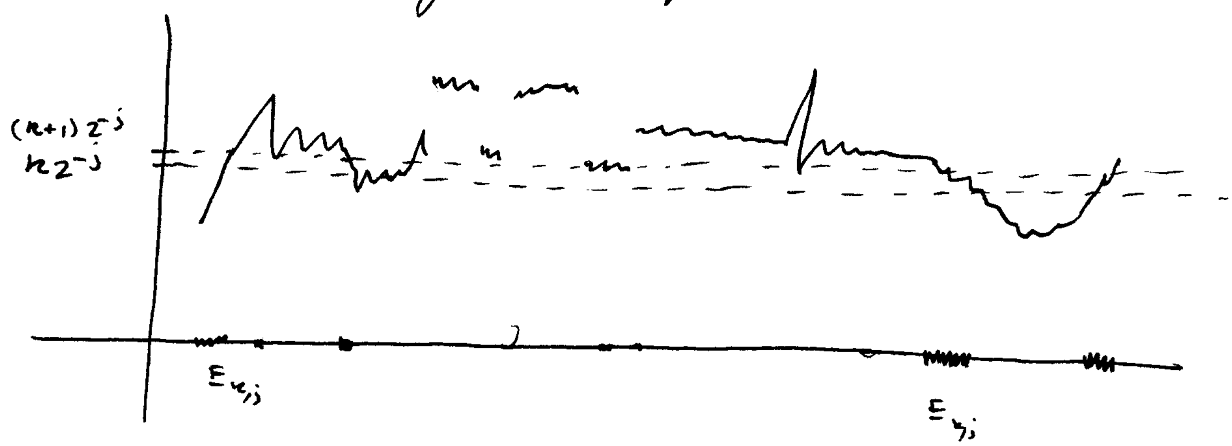
A very important example for us will be the case $F(x) = x$ which gives the classical Lebesgue measure on $\mathbb{R}$.

The Construction of the Lebesgue Measurable sets and Lebesgue measure.

We will begin this today. The construction uses the notion of "Outer measure". The abstraction of the ideas in this construction will finally lead us to the Caratheodory extension thm ((8.1.22) in CI).

<u>Integrals on Measure Spaces.</u>

Recall the approach to integration proposed by Lebesgue.



$$\int f\,dm \approx \sum_k k\cdot 2^{-j}\, m(E_{k,j})$$

In the general setting, we have a measure space $(E, \mathcal{B}, \mu)$ and we would like to "integrate" a function by some procedure like

$$\int f \, d\mu \approx \sum_k k 2^{-j} \mu\left(x \in E : \frac{k}{2^j} \leq f(x) < \frac{k+1}{2^j}\right)$$

To make sense of this in our set-up we must have

$$\left\{ x \in E : \frac{k}{2^j} \leq f(x) < \frac{k+1}{2^j} \right\} \in \mathcal{B}.$$

Given a measure space $(E, \mathcal{B}, \mu)$ we say that a real valued function $f : E \to \mathbb{R}$ is <u>measurable</u> if for any interval $I \subset \mathbb{R}$, we have $\{ x \in E : f(x) \in I \} \in \mathcal{B}$.

Instead of intervals I, we could use Borel sets A in this definition. To see this, note that

$(*)\ \{\{x \in E : f(x) \in A\} : A \text{ Borel}, A \subset \mathbb{R}\}$

is a σ-algebra on E.

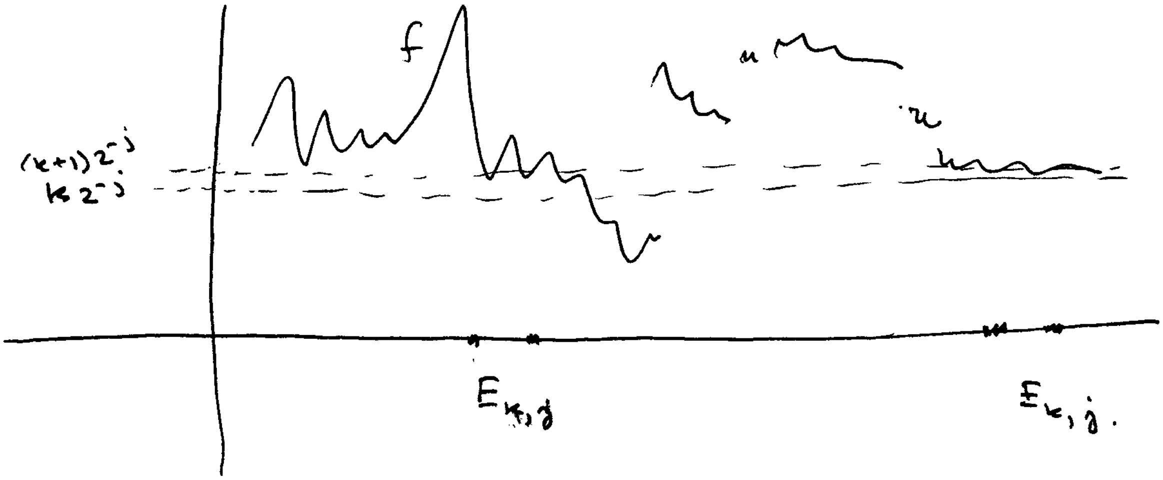
We claim that $(*)$ is generated by

$(**)\ \{\{x \in E : f(x) \in I\} : I \subset \mathbb{R}, I \text{ is an interval}\}$.

- $(**)$ is a π-system
- Let H be any λ-system containing $(**)$, then $f(H)$ is a λ-system in $\mathbb{R}$ containing all intervals and $\therefore$ all Borel sets.
- $\therefore\ \{x \in E : f(x) \in A\} \in H$ for any Borel set $A \subset \mathbb{R}$.

<u>exercise</u>:
- fill in the details
- Do exercise 3.1.10
- read section 3.2.

Recall the ~~[illegible]~~ approach to integration proposed by Lebesgue



$$\int f\,dm \approx \sum_k k\cdot 2^{-j}\, m\left(E_{k,j}\right).$$

In the general setting, we have a measure space $(E, \mathcal{B}, \mu)$ and we would like to "integrate" a function by some ~~[illegible]~~ procedure like

$$\int f\,d\mu \approx \sum_k k\cdot 2^{-j}\,\mu\left(x\in E : \frac{k}{2^j} \le f(x) < \frac{k+1}{2^j}\right)$$

To make sense of this in our set-up we must have

$$\left\{x\in E : \frac{k}{2^j} \le f(x) < \frac{k+1}{2^j}\right\} \in \mathcal{B}$$

$\{\{x\in E : f(x)\in A\} : A$ Borel, $A\subset\mathbb{R}\}$

is a σ-algebra on E.

<u>Claim:</u> It is generated by

$\{\{x\in E : f(x)\in I\}$ ~~$\in \mathbb{R}$~~ $: I\subset\mathbb{R}$ is an interval$\}$.

A π-system in E.

Let $\mathcal{A}$ be any λ-system in E containing it.

then $f(\mathcal{A})$ is a λ-system containing (all intervals) a π-system

so it contains all Borel sets.

$\therefore$ $\{x\in E : f(x)\in A\}\in\mathcal{A}$ for

any Borel set $A\subset\mathbb{R}$.

Do exercise 3.1.10. and read section 3.2.

Math 137 ~~[illegible]~~ → 9/15/08.

<u>Lemma:</u> (2.1.9). $G \subset \mathbb{R}^N$ open, $\delta > 0$ given.

G can be exactly covered by a countable collection of non-overlapping cubes Q with $\operatorname{diam}(Q) < \delta$.

- non overlapping means.

$$Q_1^\circ \cap Q_2^\circ = \phi \quad \text{for any } Q_1 \neq Q_2$$

in the covering.

- $\operatorname{diam}(Q) = \sup\{|x-y| : x \in Q, y \in Q\}$.

- a <u>cube</u> is any set of the form.

$$[a_1, b_1] \times \dots \times [a_N, b_N]$$

with $b_i - a_i = b_j - a_j \quad j = 1, \dots, N$.

<u>Pf:</u> Consider $Q_n = [0, 2^{-n}]^N$

and let $D_n = \{ \frac{\vec{k}}{2^n} + Q_n : k \in \mathbb{Z}^N \}$

be the <u>dyadic cubes of rank n</u>.

(*) • If $m \le n$, $Q \in \mathcal{D}_m$ and $Q' \in \mathcal{D}_n$ then either $Q' \subseteq Q$ or $\mathring{Q} \cap \mathring{Q}' = \emptyset$.

• if $Q \in \mathcal{D}_n$ then $diam(Q) = 2^{-n}\sqrt{N}$

Let n_0 be the smallest integer s.t.

$$2^{-n}\sqrt{N} < \delta$$

Let $\mathcal{E}_{n_0} = \bigcup_{n \ge n_0} \mathcal{D}_n$ and let

$\mathcal{C} \subset \mathcal{E}_{n_0}$ consist of all cubes $Q \in \mathcal{E}_{n_0}$

s.t a) $Q \subset G$

<u>and</u>

b) Q is not contained in any larger dyadic cube satisfying a).

Clearly each $x \in G$ is contained in a cube from $\mathcal{C}$

and if Q, Q' are distinct cubes in $\mathcal{C}$ then they are non-overlapping by (*).

The lemma shows that the Borel σ-algebra in $\mathbb{R}^N$ is generated by the collection of dyadic cubes.

- Problem 3.1.10
- Homework problems
-     Rademacher functions etc.
-     connection between 1.1.18 and 1.1.19.

A rectangle in $\mathbb{R}^N$ is any set of the form $I = [a_1, b_1] \times \dots \times [a_N, b_N]$ and $Vol(I) \equiv \prod_{j=1}^{N} (b_j - a_j)$.

<u>Def</u>: Let $\Gamma \subseteq \mathbb{R}^N$. The <u>outer measure</u> of Γ (or ~~E~~ Exterior Lebesgue measure) is

$$|\Gamma|_e \equiv \inf \left\{ \sum_{I \in \mathcal{C}} Vol(I) : \Gamma \subseteq \bigcup_{R \in \mathcal{C}} R \right\}.$$

where the infimum is over all covering $\mathcal{C}$ of Γ by a countable collection of rectangles (R).

<u>Claim</u>: the map

$$\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^N}} \longrightarrow |\Gamma|_e$$

is countably additive
(and $|\emptyset|_e = 0$).

$\overline{\mathcal{B}_{\mathbb{R}^N}}$ is the completion of $\mathcal{B}_{\mathbb{R}^N}$ described earlier.

Proving the claim requires some labor...

<u>Lemma</u>: If $\Gamma = \bigcup_{1}^{n} J_m$ where the J_m's are non-overlapping rectangles (disjoint interiors) then $|\Gamma|_e = \sum_{1}^{n} \mathrm{vol}(J_n)$.

For now, we assume the following

<u>Lemma</u> • If $\mathcal{C}$ is a non-overlapping finite collection of rectangles each of which is contained in the rectangle J, then

$$\mathrm{vol}(J) \geq \sum_{I \in \mathcal{C}} \mathrm{vol}(I).$$

• If $\mathcal{C}$ is any finite collection of rectangles and J is a rectangle which is covered by $\mathcal{C}$, then

$$\mathrm{vol}(J) \leq \sum_{I \in \mathcal{C}} \mathrm{vol}(I).$$

Assuming this, we prove the 1st lemma:

<u>Pf</u>: $|\Gamma|_e \leq \sum_{m=1}^{n} \mathrm{vol}(J_m)$

since $\{J_m\}_{m=1}^{n}$ is an admissible covering.

To get the opposite inequality we consider an arbitrary countable covering $\mathcal{C}$ of Γ and fix $\epsilon > 0$. We aim to show that

$$\sum_{m=1}^{n} vol(J_m) \leq \sum_{I \in \mathcal{C}} vol(I) + \epsilon.$$

Let $\mathcal{C} = \{I_\ell\}_{\ell=1}^{\infty}$.

Fatten up the I_ℓ. Choose I_ℓ' s.t. $I_\ell \subseteq (I_\ell')^\circ$ and

$$vol(I_\ell') \leq vol(I_\ell) + \epsilon \cdot 2^{-\ell}.$$

Γ is compact, so finitely many of the I_ℓ' cover Γ.

We may suppose

$\{I_1', \dots, I_L'\}$ covers Γ for some L

by our assumed lemma, for each m

$$\mathrm{Vol}(J_m) \le \sum_{\ell=1}^{L} \mathrm{vol}(J_m \cap I_\ell').$$

So

$$\sum_{m=1}^{n} \mathrm{vol}(J_m) \le \sum_{m=1}^{n} \sum_{\ell=1}^{L} \mathrm{vol}(J_m \cap I_\ell')$$

$$\le \sum_{\ell=1}^{L} \mathrm{vol}(I_\ell')$$

$$\le \sum_{\ell=1}^{\infty} \mathrm{vol}(I_\ell')$$

$$\le \sum_{\ell=1}^{\infty} (\mathrm{vol}(I_\ell) + \epsilon 2^{-\ell})$$

$$\le \sum_{\ell=1}^{\infty} \mathrm{vol}(I_\ell) + \epsilon.$$

Since ϵ was arbitrary, we have

$$\sum_{m=1}^{n} \mathrm{vol}(J_m) \le \sum_{\ell=1}^{\infty} \mathrm{vol}(I_\ell).$$

and since the covering $\mathcal{C}$ was arbitrary. $\sum_{m=1}^{n} \mathrm{vol}(J_m) \le |\Gamma|_e$.