# Math 137 9/8/08.

3.1.8

$$\mathcal{H} = \{\Gamma \in \mathcal{B} : \mu(\Gamma) = \nu(\Gamma)\}$$

$$\mu(E) = \nu(E).$$

if $\Gamma_1, \Gamma_2 \in \mathcal{H}$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$

$$\Rightarrow \mu(\Gamma_1 \cup \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2) = \nu(\Gamma_1) + \nu(\Gamma_2)$$
$$= \nu(\Gamma_1 \cup \Gamma_2).$$

$$\Rightarrow \Gamma_1 \cup \Gamma_2 \in \mathcal{H}.$$

if $\Gamma_1, \Gamma_2 \in \mathcal{H}$ and $\Gamma_1 \subseteq \Gamma_2$

$$\mu(\Gamma_2 \setminus \Gamma_1) = \mu(\Gamma_2) - \mu(\Gamma_1)$$
$$= \nu(\Gamma_2) - \nu(\Gamma_1) = \nu(\Gamma_2 \setminus \Gamma_1).$$

if $\{\Gamma_n\}_1^\infty \subseteq \mathcal{H}$ and $\Gamma_n \nearrow \Gamma$.

then

$\Gamma_0 = \emptyset$. $$\mu(\Gamma) = \sum_{j=1}^{\infty} \mu(\Gamma_j / \Gamma_{j-1}) = \sum_{j=1}^{\infty} \nu(\Gamma_j / \Gamma_{j-1}) = \nu($$

$\mathcal{H}$ is a $\lambda$ system containing $\mathcal{C}$, so by our Lemma, $\mathcal{B} \subseteq \mathcal{H}$.

3.1.12 $\Gamma_n$ $n \geq 1$ are sets.

$$\overline{\lim_{n\to\infty}} \Gamma_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Gamma_n$$

$$\underline{\lim_{n\to\infty}} \Gamma_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Gamma_n$$

$\overline{\lim_{n\to\infty}} \Gamma_n = \{x : x \in \Gamma_n$ infinitely many $n \in \mathbb{Z}^+\}$.

$\underline{\lim_{n\to\infty}} \Gamma_n = \{x : x \in \Gamma_n$ for all but finitely many $n \in \mathbb{Z}^+\}$.

$\underline{\lim_{n\to\infty}} \Gamma_n \subseteq \overline{\lim_{n\to\infty}} \Gamma_n$ with equality when $\{\Gamma_n\}$ is monotone.

i). $\bigcap_{n\geq m} \Gamma_n \nearrow \underline{\lim_{n\to\infty}} \Gamma_n$

$\bigcap_{n\geq m} \Gamma_n \subset \Gamma_m$ for each $m$

ii) $\bigcup_{n=m}^{\infty} \Gamma_n \searrow \overline{\lim_{n\to\infty}} \Gamma_n$

and $\Gamma_m \subset \bigcup_{n=m}^{\infty} \Gamma_n$ for each $m$

so if $\mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) < +\infty$

$$\mu\left(\bigcup_{n=m}^{\infty} \Gamma_n\right) \searrow \mu\left(\overline{\lim} \Gamma_n\right)$$

and $\mu(\Gamma_m) \leq \mu\left(\overline{\lim} \Gamma_n\right)$ for each $m$

so that $\overline{\lim_{m\to\infty}} \mu(\Gamma_m) \leq \mu\left(\overline{\lim} \Gamma_n\right)$.

iii). If $\underline{\lim_{n\to\infty}} \Gamma_n = \overline{\lim_{n\to\infty}} \Gamma_n$

and $\mu\left(\bigcup_1 \Gamma_n\right) < +\infty$

then.

$$\overline{\lim_{n\to\infty}} \mu(\Gamma_n) \leq \mu\left(\overline{\lim_{n\to\infty}} \Gamma_n\right) = \mu\left(\underline{\lim_{n\to\infty}} \Gamma_n\right) \leq \underline{\lim_{n\to\infty}} \mu$$

and $\therefore$ $\lim_{n\to} \mu(\Gamma_n) = \mu\left(\lim_{n\to\infty} \Gamma_n\right)$

iv). if $\sum_1^\infty \mu(\Gamma_n) < +\infty$ then

$$\mu(\overline{\lim_{n\to\infty}} \Gamma_n) \leq \mu\left(\bigcup_{n=m}^{\infty} \Gamma_n\right) \quad \text{for each } m$$

$$\leq \sum_{n=m}^{\infty} \mu(\Gamma_n)$$

and the convergence of the series implies

$$\sum_{n=m}^{\infty} \mu(\Gamma_n) \to 0 \quad \text{as } m \to \infty.$$

so $\mu(\overline{\lim_{n\to\infty}} \Gamma_n) = 0$.

## Independence

Let $(\Omega, \mathcal{F}, P)$ be a Prob. space.

for each $i \in I$ → an index set

$\mathcal{F}_i$ is a $\sigma$-algebra.

Def: The $\mathcal{F}_i$ are mutually $P$-independent iff for any finite $\{i_1, \dots i_n\} \subset I$ and choice $A_{i_m} \in \mathcal{F}_{i_m}$ $1 \le m \le n$. we have

$$P(A_{i_1} \cap \dots \cap A_{i_n}) = P(A_{i_1}) \dots P(A_{i_n}).$$

Def: If $\{A_i : i \in \mathcal{U}\}$ is a family of sets from $A$, we say the $A_i$ are independent (mutually $P$ independent) iff the $\sigma$-algebras $\mathcal{F}_i = \{\phi, A_i, A_i^c, \Omega\}$ are e.g. two sets $A_1, A_2$ are ind. iff

$$P(A_1 \cap A_2) = P(A_1) P(A_2).$$

We just need to check what happens with $A_1^c \cap A_2^c = (A_1 \cup A_2)^c$.

$$\begin{aligned} P(A_1^c \cap A_2^c) &= 1 - P(A_1 \cup A_2) \\ &= 1 - \big(P(A_1) + P(A_2) - P(A_1 \cap A_2).\big) \\ &= 1 - P(A_1) - P(A_2) + P(A_1) P(A_2). \\ &= (1 - P(A_1))(1 - P(A_2).) \end{aligned}$$

$$\big(P(A_1 \cap A_2^c) = P(A_1) - P(A_1 \cap A_2) = P(A_1)(1 - P(A_2))$$

etc...

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## Kolmogorov's 0-1 law

Note that, by exercise 1.1.12, to check independence of $\sigma$-algebras, it is enough to check the definition for generating $\pi$-systems $\mathcal{C}_i \subset \mathcal{F}_i$.

Let $\mathcal{U}$ be an index set.

$$\mathcal{F}_\emptyset = \{\emptyset, \Omega\}.$$

and for $\Lambda \subseteq \mathcal{U}$ $\Lambda \neq \emptyset$

$$\mathcal{F}_\Lambda = \underbrace{\bigvee_{i \in \Lambda} \mathcal{F}_i}$$

the σ-alg. generated by the $\bigcup_{i \in \Lambda} \mathcal{F}_i$.

<u>Def</u>: The <u>tail σ-algebra</u> $\mathcal{T}$ is $\displaystyle\bigcap_{\Lambda \text{ Finite}} \mathcal{F}_{\Lambda^c}$

e.g. If $\mathcal{U}$ is finite

$$\mathcal{T} = \{\emptyset, \Omega\}.$$

and $P(A) \in \{0, 1\}$ for each $A \in \mathcal{T}$.

Claim:

If the $\mathcal{F}_i$ are independent then this is true for any $\mathcal{C}$.

1st $\mathcal{F}_{F_1}$ is independent of $\mathcal{F}_{F_2}$

when $F_1$ and $F_2$ are finite and ~~distinct~~. disjoint.

This is because

$$\{A : A = A_{i_1} \cap \dots \cap A_{i_m},\ i_j \in F_1 \quad j = 1 \dots m\}.$$

is a generating $\pi$-system for $\mathcal{F}_{F_1}$

and $\{B : B = B_{j_1} \cap \dots \cap B_{j_p},\ j_n \in F_2,\ n = 1, \dots, p\}$

is a generating $\pi$-system for $\mathcal{F}_{F_2}$.

(use exercise 1.12)

Now, for <u>any</u> $\Lambda \subset \mathcal{U}$

(*) $\quad \bigcup \{ \mathcal{F}_F : F \text{ is a finite subset of } \Lambda \}$

is an algebra. (check it).

and this algebra generates $\mathcal{F}_\Lambda$

By exercise 1.12 (again)

- $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\Lambda^c}$ are $P$-independent

for every $\Lambda \subset \mathcal{U}$

- $\mathcal{T}$ is $P$ independent of $\mathcal{F}_F$

where $F$ is any finite

subset of $\cancel{\mathcal{F}}\mathcal{U}$

$\therefore$ $\mathcal{T}$ is ind of $\bigcup \{\cancel{\mathcal{F}_F}\} \mathcal{F}_{\mathcal{U}}$.

(ind of any el of (*)).

Since $\mathcal{T} \subseteq \mathcal{F}_{\mathcal{U}}$

$\mathcal{T}$ is independent of itself

So $P(A \cap B) = P(A) P(B) \quad \forall A, B \in \mathcal{T}$

but then $P(A) = P(A \cap A) = P(A)^2$

so $P(A) \in \{0, 1\}$.

e.g. $\{A_n\}_{n=1}^{\infty} \subset \Omega$

$$\overline{\lim_{n\to\infty}} A_n \equiv \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n$$

$$= \{\omega : \omega \in A_n \text{ for infinitely many } n \in \mathbb{Z}^+\}.$$

is msble w.r.t the tail algebra determined by $\mathcal{F}_n = \{\emptyset, A_n, A_n^c, \Omega\}$.

So if $A_n$'s are P-ind. then

$$P(\overline{\lim_{n\to\infty}} A_n) \in \{0, 1\}.$$

We saw that $\left( \{A_n : n \in \mathbb{Z}^+\} \subseteq \mathcal{F} \right)$.

$$\sum_{n=1}^{\infty} P(A_n) < \infty \implies P\left( \overline{\lim_{n\to\infty}} A_n \right) = 0.$$

Conversely, if the $A_n$'s are $P$-independent sets, then

$$\sum_{n=1}^{\infty} P(A_n) = \infty \implies P\left( \overline{\lim_{n\to\infty}} A_n \right) = 1.$$

<u>Pf:</u> $$\left(\overline{\lim_{n\to\infty}} A_n\right)^c = \left( \bigcap_{m=1}^{\infty} \bigcup_{n\ge m} A_n \right)^c = \bigcup_{m=1}^{\infty} \left( \bigcup_{n\ge m} A_n \right)^c$$

$$= \bigcup_{m=1}^{\infty} \left( \bigcap_{n \ge m} A_n^c \right)$$

so $P\left( \left(\overline{\lim} A_n\right)^c \right) = 0$ iff $P\left( \bigcap_{n\ge m} A_n^c \right) = 0$ for each $m$.

But

$$P\left( \bigcap_{n\ge m} A_n^c \right) = \lim_{N\to\infty} \prod_{n=m}^{\infty} (1 - P(A_n))$$

$$\le \lim_{N\to\infty} e^{-\sum_{n=m}^{N} P(A_n)} = 0.$$

since $\sum_{n=1}^{\infty} P(A_n) = +\infty$. $\quad (1 - t \le e^{-t})$ $t \in [0,$

The current assignment is due next Monday.

Add 1.1.18 1.1.19 to it. in [PT]