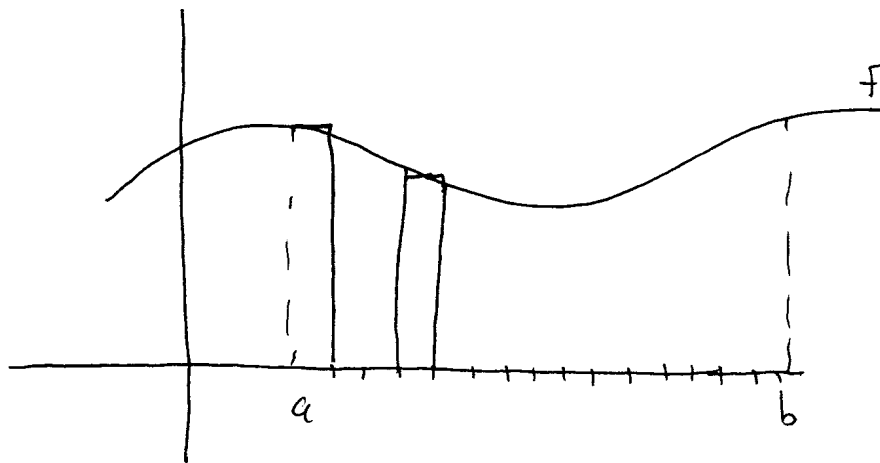


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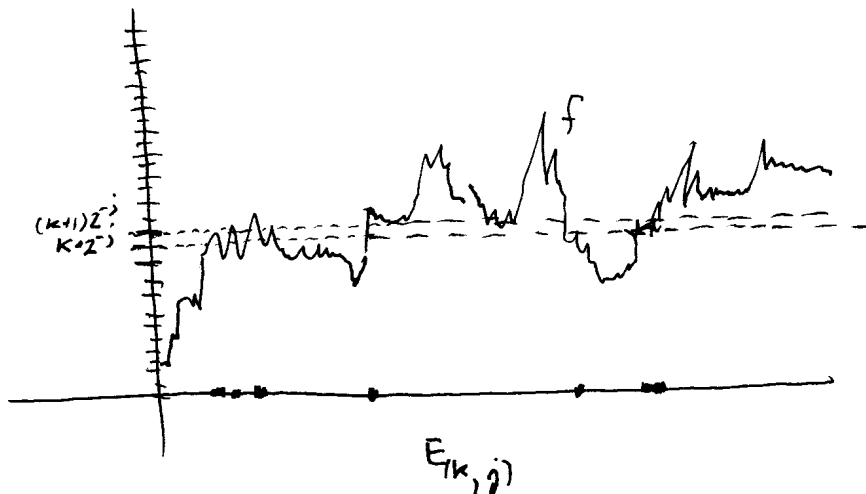
## Riemann Integration



partition the domain  
approximate  $f$  in each partition by its  
value at some point.

$\int_a^b f$  is given as a limit of  
integrals of piecewise constant  
functions.

Lebesgue I integration



$\int f$  is given as a limit  
of integrals of "simple functions".

$$\int f \approx \sum_k (k \cdot 2^{-j}) m(E_{k,j})$$

where  $m(E_{k,j})$  is the "measure" of  
 $E_{k,j}$

If  $f$  is complicated, then the sets  
 $E_{k,j}$  may be complicated.

What will we mean by their  
"measure"?

Lebesgue discovered that there is a  
class of subsets of  $\mathbb{R}^n$  which is  
sufficiently well behaved so that  
we can define a natural measure,  
but sufficiently general so that

we can integrate a much wider class of functions than the Riemann integrable functions.

What kind of special subsets was Lebesgue after? What properties should a measure possess to make the integration program work?

We will look ahead now and consider these concepts in an abstract but ultimately very useful form.

Let  $E$  be a set.

def:  $\mathcal{P}(E) = \{ \Gamma : \Gamma \subseteq E \}$ .

"the power set of  $E$ ".

def: An algebra over  $E$   
is a subset of  $\mathcal{P}(E)$   
 $\mathcal{A} \subset \mathcal{P}(E)$  s.t.

$$i) \quad \emptyset \in \mathcal{A}$$

$$ii) \quad P \in \mathcal{A} \Rightarrow P^c \in \mathcal{A}$$

$$iii) \quad P_1, P_2 \in \mathcal{A} \Rightarrow P_1 \cup P_2 \in \mathcal{A}.$$

$$\text{Since } (P_1 \cup P_2)^c = P_1^c \cap P_2^c,$$

$\mathcal{A}$  is also closed under intersections.

$$\therefore \text{ if } A, B \in \mathcal{A} \text{ then } A \cap B = A \cap B^c \in \mathcal{A}$$

By induction,  $\mathcal{A}$  is closed under  
finite unions and intersections.

Def: A  $\sigma$ -algebra over  $E$  is  
an algebra  $\mathcal{B} \subseteq \mathcal{P}(E)$  which  
is closed under countable unions.

i.e. if  $A_1, \dots, A_n, \dots$   $n=1, 2, 3, \dots$

are sets in  $\mathcal{B}$  then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}.$$

Note that  $\left(\bigcup_{i=1}^{\infty} P_i^c\right)^c = \bigcap_{i=1}^{\infty} P_i$ , so

$\mathcal{B}$  is also closed under countable  
intersections.

The trivial  $\sigma$ -algebras ( $\sigma$ -algebras)

are  $\{\emptyset, E\}$

and  $\mathcal{P}(E)$

Lemma: • Any intersection of algebras is an algebra

Pf: i)  $\emptyset$  is in the intersection  
since it is in each algebra.

ii) if  $\Gamma$  is in the intersection  
then  $\Gamma$  is in each algebra  
 $\therefore \Gamma^c$  is in each algebra  
and  $\Gamma^c$  is in the intersection.

iii) if  $\Gamma_1, \Gamma_2$  are in the  
intersection then  $\Gamma_1 \cup \Gamma_2$  is  
in each algebra.

• Any intersection of  $\sigma$ -algebras  
is a  $\sigma$ -algebra.

Pf: if  $\Gamma_i \quad i=1, 2, 3, \dots$   
are in each  $\sigma$ -algebra  
then  $\bigcup_{i=1}^{\infty} \Gamma_i$  is in each  $\sigma$ -algebra.

The lemma shows that any  $C \subset \mathcal{P}(E)$  is contained in a unique minimal algebra  $\mathcal{A}(E; C)$  and a unique minimal  $\sigma$ -algebra  $\sigma(E; C)$  over  $E$ .

We say that  $\underline{C}$  generates  $\sigma(E; C)$ .

e.g. If  $E$  is a topological space and  $\mathcal{O}$  is the collection of open sets in  $E$  then  $\sigma(E, \mathcal{O})$  is called the Borel  $\sigma$ -algebra on  $E$ . and denoted  $\mathcal{B}_E$ . Elements of  $\mathcal{B}_E$  are called Borel measurable.

e.g. For each  $t \in [0, +\infty)$ , suppose

$X_t: \Omega \rightarrow \mathbb{R}$  is a function on some (probability space) space  $\Omega$ .

$$X_t(\omega) \in \mathbb{R} \quad \text{for } \omega \in \Omega$$

$\mathcal{F}_s = \sigma(X_t; t \leq s)$  is the  $\sigma$ -algebra (on  $\Omega$ ) generated by sets of the form

$$\{\omega: X_{t_j}(\omega) \in A\} \quad \text{where } A \in \mathcal{B}_{\mathbb{R}}, \\ t_j \in \mathbb{R}^+.$$

Think of  $X_t(\omega)$  as a random process whose trajectory (in  $t$  = time) depends on the random choice of  $\omega$ . The  $\sigma$ -algebras  $\mathcal{F}_s$  are a model of the notion of "information about the process" which is available by time  $= s$ .

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$\pi$ -systems and  $\lambda$ -systems

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Def: A  $\pi$ -system is a subcol of  $\mathcal{P}(E)$   
 $C \subseteq \mathcal{P}(E)$  s.t.

$$A_1, A_2 \in C \Rightarrow A_1 \cap A_2 \in C.$$



"Most" useful generating subsets of  $\sigma$ -algebras are  $\pi$ -systems.

e.g.

- open subsets of a topological space
- rectangles in  $\mathbb{R}^N$   
 $[a_1, b_1] \times \dots \times [a_N, b_N]$

Def:  $\mathcal{H} \subseteq \mathcal{P}(E)$  is a  $\lambda$ -system over  $E$

if

i)  $E \in \mathcal{H}$

ii)  $P_1, P_2 \in \mathcal{H}$  and  $P_1 \cap P_2 = \emptyset \Rightarrow P_1 \cup P_2 \in \mathcal{H}$

iii)  $P_1, P_2 \in \mathcal{H}$  and  $P_1 \subseteq P_2 \Rightarrow P_2 \setminus P_1 \in \mathcal{H}$

iv)  $\{P_n\} \subseteq \mathcal{H}$  and  $P_n \nearrow P \Rightarrow P \in \mathcal{H}$ .  
( $P_j \subseteq P_{j+1}$  for each  $j$  and  $P = \bigcup_{j=1}^{\infty} P_j$ )

The properties defining a  $\lambda$ -system are exactly the ones which turn a  $\pi$ -system into a  $\sigma$ -algebra.

Lemma:

1. any intersection of  $\left\{ \begin{array}{l} \pi\text{-systems} \\ \lambda\text{-systems} \end{array} \right\}$  is a  $\left\{ \begin{array}{l} \pi\text{-system} \\ \lambda\text{-system} \end{array} \right\}$ .
2.  $\mathcal{B} \subset \mathcal{P}(E)$  is a  $\sigma$ -algebra over  $E$   
iff  $\mathcal{B}$  is both a  $\pi$ -system and a  $\lambda$ -system.
3. If  $\mathcal{C} \subset \mathcal{P}(E)$  is a  $\pi$ -system, then  
 $\sigma(E; \mathcal{C})$  is the smallest  $\lambda$ -system  
over  $E$  containing  $\mathcal{C}$ .

Pf: 1. is easily checked using the definitions.  
2. It's clear that any  $\sigma$ -algebra  
is a  $\pi$ -system and a  $\lambda$ -system.

Ulf  $\mathcal{B}$  is a  $\pi$ -system and a  $\lambda$ -system over  
 $E$  then  $E \in \mathcal{B}$  and  $E \setminus A \in \mathcal{B}$   
for each  $A \in \mathcal{B}$  so  $E \setminus E = \emptyset \in \mathcal{B}$   
and  $A^c = E \setminus A \in \mathcal{B}$  for each  $A \in \mathcal{B}$ .

If  $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{B}$  then

$$\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}_1 \cup (\mathcal{P}_2 \setminus \mathcal{P}_1) \in \mathcal{B}.$$



So far, we have verified that  $B$  is an algebra.

if  $P_1, \dots, P_n, \dots \in B$ .

then  $A_n = \bigcup_{i=1}^n P_i \in B$  and

$A_n \nearrow \bigcup_{i=1}^{\infty} P_i$ , so  $\bigcup_{i=1}^{\infty} P_i \in B$ .

$\therefore B$  is a  $\sigma$ -algebra.

3. Let  $C$  be a  $\pi$ -system and let  $H$  be the smallest  $\sigma$ -system over  $E$  containing  $C$ . Since  $\mathcal{O}(E, C)$  is a  $\sigma$ -system <sup>containing  $C$</sup> ,  $\mathcal{O}(E, C) \supseteq H$ .

If we can show that  $H$  is a  $\pi$ -system then by 2)  $H$  is a  $\sigma$ -algebra. and therefore  $H = \mathcal{O}(E, C)$  since  $\mathcal{O}(E, C)$  is the smallest  $\sigma$ -algebra containing  $C$ .

In other words, we would like to show that

→  $H \subset H_2 \equiv \{P \subseteq E : P \cap \Delta \in H \text{ for all } \Delta \in \mathcal{C}\}$   
 (and this will follow if  $H_2$  is a  $\lambda$ -system containing  $\mathcal{C}$ ).

First consider

$$H_1 \equiv \{P \subseteq E : P \cap \Delta \in H \text{ for all } \Delta \in \mathcal{C}\}$$

Claim:  $H_1$  is a  $\lambda$  system.

$$i) E \cap \Delta = \Delta \quad \forall \Delta \in \mathcal{C} \subset H_1 \\ \text{so } E \in H_1$$

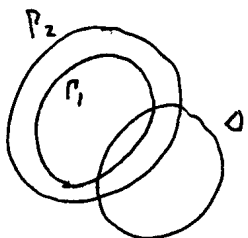
$$ii) P_1, P_2 \in H_1, \quad P_1 \cap P_2 \neq \emptyset$$

$$(\text{since } H \text{ is a } \lambda\text{-system}) \Rightarrow P_1 \cup P_2 \cap \Delta = (P_1 \cap \Delta) \cup (P_2 \cap \Delta) \in H \text{ for } \Delta \in \mathcal{C}.$$

where the union on the right is disjoint.

$$iii) \text{ if } P_1, P_2 \in H_1, \quad P_1 \subseteq P_2,$$

$$\text{then for any } \Delta \in \mathcal{C}, P_2 \cap P_1^c \cap \Delta = (P_2 \cap \Delta) \cap (P_1 \cap \Delta)^c \in H.$$



$$\Rightarrow P_2 \setminus P_1 \in H_1. \quad (\text{since } P_1 \cap \Delta \subset P_2 \cap \Delta \text{ and both are in } H)$$

iv) if  $\Gamma_j \supset \Gamma$  and  $\Gamma_j \in \mathcal{H}_1$  for each  $j$

then  $\Gamma_j \cap \Delta \supset \Gamma \cap \Delta$  for each  $\Delta \in \mathcal{C}$

so  $\Gamma \cap \Delta \in \mathcal{H}$  for each  $\Delta \in \mathcal{C}$ .

$\therefore \Gamma \in \mathcal{H}_1$ .

Now since  $\mathcal{C}$  is a  $\pi$ -system

$\mathcal{C} \in \mathcal{H}_1$  and  $\therefore$

$\mathcal{H} \subset \mathcal{H}_1$  since  $\mathcal{H}$  is the smallest  $\lambda$  system containing  $\mathcal{H} \cup \mathcal{C}$ .

\*  $\left\{ \begin{array}{l} \text{We have now shown that} \\ \text{for any } \Gamma \in \mathcal{H} \quad \Gamma \cap \Delta \in \mathcal{H} \quad \forall \Delta \in \mathcal{C}. \end{array} \right.$

Look again at  $\mathcal{H}_2$

$$\mathcal{H}_2 = \{ \Gamma \subseteq E : \Gamma \cap \Delta \in \mathcal{H} \quad \forall \Delta \in \mathcal{H} \}.$$

Reversing the roles of  $\Gamma, \Delta$  in  $\mathcal{H}$ , we see that

$$\mathcal{C} \subset \mathcal{H}_2$$

and we check, in exactly the same way as above, that  $\mathcal{H}_2$  is a  $\lambda$ -system.

$\therefore H \subset H_2$  (since  $H$  is the smallest  $\sigma$ -system containing  $C$ ).



Def: A measurable space is a pair  $(E, \mathcal{B})$  where  $E$  is a set and  $\mathcal{B}$  is a  $\sigma$ -algebra over  $E$ .

Def: A measure on the measurable space  $(E, \mathcal{B})$  is a map  $\mu: \mathcal{B} \rightarrow [0, \infty]$  s.t.

i)  $\mu(\emptyset) = 0$ .

$\mu$  is  
(countably  
additive)

ii)  $\{P_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$  and  $P_m \cap P_n = \emptyset$   $m \neq n$   
 $\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} P_n\right) = \sum_{n=1}^{\infty} \mu(P_n)$ .

if  $\mu(E) < +\infty$ ,  $\mu$  is called a finite measure.

if  $\mu(E) = 1$ ,  $\mu$  is called a probability measure.

if  $(E, \mathcal{B})$  is a measurable space and  $\mu$  a measure on  $(E, \mathcal{B})$

then  $(E, \mathcal{B}, \mu)$  is called a measure space or  $\begin{cases} \text{finite measure space} \\ \text{or} \\ \text{probability space} \end{cases}$

if  $\begin{cases} \mu \text{ is finite} \\ \mu(E) = 1 \end{cases}$

Def. Lemma: Theorem:

Let  $(E, \mathcal{B}, \mu)$  be a measure space.

1). If  $P_1, P_2 \in \mathcal{B}$  and  $P_1 \subseteq P_2$

then  $\mu(P_1) \leq \mu(P_2)$  and

when  $\mu(P_1) < +\infty$ , then

$$\mu(P_2 \setminus P_1) = \mu(P_2) - \mu(P_1)$$

2). If  $\{\Gamma_n\}_{n=1}^{\infty} \subset \mathcal{B}$  then

$$a) \quad \Gamma_n \nearrow \Gamma \Rightarrow \mu(\Gamma_n) \nearrow \mu(\Gamma)$$

$$b) \quad \Gamma_n \searrow \Gamma \text{ and } \mu(\Gamma_1) < +\infty \\ \Rightarrow \mu(\Gamma_n) \searrow \mu(\Gamma)$$

$$c) \quad \mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) \leq \sum_{n=1}^{\infty} \mu(\Gamma_n)$$

$$d) \quad \mu(\Gamma_m \cap \Gamma_n) = 0, m \neq n$$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(\Gamma_n) = \mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right).$$

Pf:

$$1) \quad \Gamma_2 = \Gamma_1 \cup (\Gamma_2 \setminus \Gamma_1) \text{ if } \Gamma_1 \subseteq \Gamma_2$$

$$\text{so } \mu(\Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2 \setminus \Gamma_1)$$

$$\therefore \mu(\Gamma_1) \leq \mu(\Gamma_2).$$

and if  $\mu(\Gamma_1) \neq +\infty$  then

$$\mu(\Gamma_2) - \mu(\Gamma_1) = \mu(\Gamma_2 \setminus \Gamma_1).$$

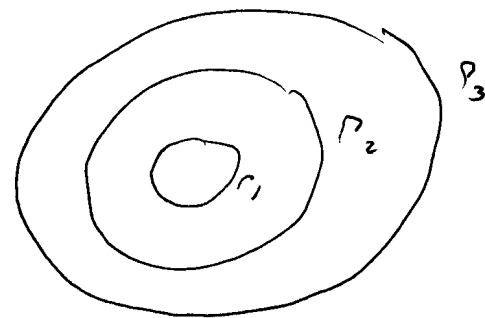


2) a) if  $\{P_n\}_{n=1}^{\infty} \subset \mathcal{B}$  and

$$P_n \nearrow P$$

put  $P_0 = \emptyset$  and

$$A_{n+1} = P_{n+1} \setminus P_n$$



$$n \geq 0.$$

then  $A_m \cap A_n = \emptyset$  for  $m \neq n$

$$\text{and } P_n = \bigcup_{i=1}^n A_i \quad \text{so } P = \bigcup_{i=1}^{\infty} A_i.$$

It follows that

$$\mu(P_n) = \sum_{i=1}^n \mu(A_i) \nearrow \sum_{i=1}^{\infty} \mu(A_i) = \mu(P).$$

b)

$$P_1 \setminus P_n \nearrow P_1 \setminus P$$

$$\text{so } \mu(P_1) - \mu(P_n) \nearrow \mu(P_1) - \mu(P).$$

If  $\mu(P_1) < +\infty$ , we can subtract it from both sides to get

$$\mu(P_n) \searrow \mu(P).$$

c). Put  $P_0 = \emptyset$

$$A_{n+1} = P_{n+1} \setminus \bigcup_{m=1}^n P_m \quad n \geq 0.$$

$$\text{Then } P_n = A_n \cup \left( P_n \cap \bigcup_{m=1}^{n-1} P_m \right)$$

$$= A_n \cup \bigcup_{m=1}^{n-1} (P_m \cap P_n)$$

$$\text{and } \bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} A_n.$$

Since  $A_m \cap A_n \neq \emptyset$  for  $m \neq n$ .

$$\begin{aligned} \mu \left( \bigcup_{n=1}^{\infty} P_n \right) &= \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n) \\ &\leq \sum_{n=1}^{\infty} \mu(P_n) \leq \sum_{n=1}^{\infty} \mu(A_n) + \mu \left( \bigcup_{m=1}^{n-1} P_m \cap P_n \right) \end{aligned}$$

This proves c) and for d)

notice that equality holds through

$$\text{if } \sum_{n=1}^{\infty} \mu \left( \bigcup_{m=1}^{n-1} P_m \cap P_n \right) = 0.$$

$$\text{But } \mu \left( \bigcup_{m=1}^{n-1} P_m \cap P_n \right) \leq \sum_{m=1}^{n-1} \mu(P_m \cap P_n) = 0$$

$$\text{if } \mu(P_m \cap P_n) = 0 \quad \forall m \neq n.$$



For next time

Exercise

Given a measure space  $(E, \mathcal{B}, \mu)$ ,

consider the class of sets  $\mathcal{P} \subseteq E$

s.t.  $\exists A, B \in \mathcal{B}$  with

$$A \subseteq \mathcal{P} \subseteq B \quad \text{and} \quad \mu(B \setminus A) = 0.$$

Show that this class, denoted  $\bar{\mathcal{P}}^\mu$  is a  $\sigma$ -algebra.

Read 3.1, 1.1 and 1.2 in [CI]  
and 1.1 in [PT].

Try the 1st two exercises in  
each section (assigned) in [CI]  
without reading the solution.

Do exercises 1.1.10 and 1.1.12  
in [PT].