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Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities

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Abstract

A maximizing function, f, is shown to exist for the HLS inequality on $\mathbf{R}^n$: $\| |x|^{-\lambda} * f \|_q \le N_{p,\lambda,n} \|f\|_p$, with N being the sharp constant and $1/p + \lambda/n = 1 + 1/q$, $1 < p, q, n/\lambda < \infty$. When $p = q'$ or $p = 2$ or $q = 2$, f and N are explicitly evaluated. A maximizing f is also shown to exist for other inequalities:

(i) The Okikiolu, Glaser, Martin, Grosse, Thirring inequality: $K_{n,p} \| \nabla f \|_2 \ge \| |x|^{-b} f \|_p$, $n \ge 3$, $0 \le b < 1$, $p = 2n/(2b + n - 2)$. (This was known before, but the proof here has certain simplifications.)

(ii) The doubly weighted HLS inequality of Stein and Weiss:

$$\left\| \int V(x, y) f(y)\, dy \right\|_q \le P_{\alpha,\beta,p,\lambda,n} \|f\|_p$$

with $V(x, y) = |x|^{-\beta} |x - y|^{-\lambda} |y|^{-\alpha}$, $0 \le \alpha < n/p'$, $0 \le \beta < n/q$, $1/p + (\lambda + \alpha + \beta)/n = 1 + 1/q$.

(iii) The weighted Young inequality: $\| |x|^{\gamma} f \|_p^m \ge Q_{p,m,n} \| f^{(m)} \|_\infty$, where $f^{(m)}(x)$ is the m-fold convolution of f with itself, $m \ge 3$, $m/(m-1) \le p < m$, $\gamma/n + 1/p = (m-1)/m$. When $p = m/(m-1)$ or $p = 2$, f and Q are explicitly evaluated.

I. Introduction

A classical inequality, due to Hardy and Littlewood [15], [16] and Sobolev [26] (see also [11]) states that

$$(1.1) \qquad \left| \int\!\!\int f(x) |x - y|^{-\lambda} g(y)\, dx\, dy \right| \le N_{p,\lambda,n} \|f\|_p \|g\|_t$$

for all $f \in L^p(\mathbf{R}^n)$, $g \in L^t(\mathbf{R}^n)$, $1 < p,\ t < \infty$, $1/p + 1/t + \lambda/n = 2$ and $0 < \lambda < n$. (By notational definition, $N_{p,\lambda,n}$ is the best, or sharp constant in (1.1).)

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The main purpose of this paper is two-fold. In Section II, it is shown that a maximizing pair f, g exists for (1.1), i.e., a pair that gives equality in (1.1). This will require the use of two rearrangement inequalities and a new compactness technique (Lemma 2.7) for maximizing sequences. From the point of view of general methodology, this is perhaps the most interesting part of this work. In Section III, N and f, g are explicitly computed for the case $t = p$ and, as a corollary, for the cases $t = 2$ or $p = 2$. This part is amusing for the following reason: one can guess what the pair f, g ought to be and verify that they satisfy the integral equations (Euler-Lagrange equations) for (1.1). But these equations are nonlinear and it is far from clear that this choice is actually maximal. The proof that it is so requires use of stereographic projection from $\mathbf{R}^n$ to the sphere $\mathbf{S}^n$ and exploitation of the symmetry of f, g given by the rearrangement inequalities used in Section II.

Additional examples are given of the techniques of Section II. Section IV contains a comparatively simple proof of the existence of a maximizing f for the Sobolev inequality

$$K_n \|\nabla f\|_2 \geq \|f\|_{2n/(n-2)}, \qquad n \geq 3 \tag{1.2}$$

and its generalization due to Okikiolu [21], Glaser, Martin, Grosse and Thirring [14]:

$$K_{n,p} \|\nabla f\|_2 \geq \| |x|^{-b} f \|_p, \qquad n \geq 3 \tag{1.3}$$

for $0 \leq b < 1$ and $p = 2n/(2b + n - 2)$. Of course (1.2) has been treated before by Aubin [2] and Talenti [31] and (1.3) in [14], but the directness of the proofs given here may be of some value.

Section V uses the techniques of Section II to prove the existence of a maximizing f, g for the doubly weighted HLS inequality of Stein and Weiss [29]:

$$\left| \int\int g(x) V(x, y) f(y)\, dx\, dy \right| \leq P_{\alpha,\beta,p,\lambda,n} \|f\|_p \|g\|_t \tag{1.4}$$

with $V(x, y) = |x|^{-\beta} |x - y|^{-\lambda} |y|^{-\alpha}$, $0 \leq \alpha < n/p'$, $0 \leq \beta < n/t'$, $1/p + 1/t + (\lambda + \alpha + \beta)/n = 2$. Finally, the weighted Young inequality is shown to have a maximizing f:

$$Q_{p,m,n} \|f^{(m)}\|_\infty \leq \| |x|^\gamma f \|_p^m, \qquad m \geq 3 \tag{1.5}$$

where $f^{(m)}(x)$ is the m-fold convolution of f with itself and $m/(m - 1) \leq p < m$, $\gamma/n + 1/p = (m - 1)/m$. Moreover, Q can be evaluated in two cases: $p = m/(m - 1)$ and $p = 2$. The latter case turns out, by Fourier transformation, to be (1.1) in disguise with $p = t$. Thus, the evaluation of the sharp constant in (1.5) for $p = 2$ brings the work full circle.

My indebtedness to Alan Sokal is profound. He stimulated this investigation by suggesting that (1.5) was true for $m = 4$, $p = 2$, a case which arose in his study of quantum field theory [27]. Later he proposed the general case of (1.5). He also suggested that the techniques of Section II would work for (1.4). Throughout the course of this work he was a constant source of encouragement and stimulation. I am also indebted to Henri Berestycki for his encouragement. I thank Haïm Brezis for pointing out the last part of Lemma 2.7 and I thank the referee for many helpful remarks, in particular for drawing my attention to [3]. I am most grateful to the Institute for Advanced Study for its support and hospitality.

A technical remark can be made about (1.1) in the context of *weak* L^p *spaces*, $|x|^{-\lambda} \in L_w^{n/\lambda}(\mathbf{R}^n)$. There are two definitions of what is meant by $\|h\|_{q,w}$ for $1 < q < \infty$. One is

$$(1.6)\qquad \|h\|_{q,w}^{*} = (n/\sigma_n)^{1/q} \sup_{\alpha>0} \alpha\mu\{x \mid |h(x)| > \alpha\}^{1/q}$$

where σ_n is the area of the unit sphere, (2.13), and μ is Lebesgue measure. This is not a true norm (the triangle inequality is not satisfied), but it is convenient and it is equivalent to the following, due to Calderón, which is a true norm:

$$(1.7)\qquad \|h\|_{q,w}^{**} = (1/q')(n/\sigma_n)^{1/q} \sup_A \mu(A)^{-1/q'} \int_A |h(x)|\, dx$$

for $0 < \mu(A) < \infty$. Clearly $|x|^{-\lambda}$ has unit norm in both definitions ($q = n/\lambda$). The generalization of (1.1) is

$$(1.8)\qquad \left|\int\int f(x)h(x-y)g(y)\,dx\,dy\right| \le N_{p,\lambda,n}\|f\|_p\|g\|_t\|h\|_{q,w}$$

with $q = n/\lambda$, $1/p + 1/t + 1/q = 2$, $1 < p, t, q < \infty$, and *the same* $N_{p,\lambda,n}$ is sharp in (1.8) as in (1.1). *Either one* of the two definitions, (1.6) or (1.7), may be used in (1.8), and the same N is sharp for both.

The justification for (1.8) is the following: if we replace f, g, h by their symmetric decreasing rearrangement, f^*, g^*, h^* (see Section II), the left side of (1.8) does not decrease. All the norms on the right side of (1.8) are invariant. The maximizing f, g for (1.1) satisfies $f = f^*$, $g = g^*$ (Section II). We note that $|x|^{-\lambda} = \sup\{h(x) \mid \|h\|_{p,w}^{*} \le 1, h = h^*\}$. Thus, (1.8) holds with (1.6). The proof for (1.7) is trickier. Let $b = f * g$. Clearly, $b = b^*$ since $f = f^*$, $g = g^*$. Thus, $b(x) = \int_0^\infty da\, \chi_a(x)$ where χ_a is the characteristic function of $T_a = \{x \mid b(x) > a\}$, which is a ball of some radius R_a. Assume that $\|h\|_{q,w}^{**} = 1$. The left side of

(1.8) is

$$\int b(x)h(x)\,dx = \int_0^\infty da \int \chi_a(x)h(x)\,dx \le \int_0^\infty da\, q'(\sigma_n/n)^{1/q}|T_a|^{1/q'}$$
$$= \int_0^\infty da\, q'(\sigma_n/n)R_a^{n-\lambda} = \int_0^\infty da \int \chi_a(x)|x|^{-\lambda}\,dx$$
$$= \int b(x)|x|^{-\lambda}\,dx.$$

II. Existence of a maximizing function

Here we shall establish the existence of a maximizing pair of functions f, g giving equality in (1.1). This means finding $f \in L^p$, $p^{-1} + \lambda/n = 1 + q^{-1}$, such that if

(2.1) $$R(F) \equiv \||x|^{-\lambda} * F\|_q / \|F\|_p, \qquad F \neq 0, \text{ then}$$

(2.2) $$R(f) = N_{p,\lambda,n} \equiv \sup\{R(F) | F \in L^p, F \neq 0\}.$$

Some remarks might be helpful to explain the difficulties to be faced in finding this f. First, the usual way to find f is by a compactness argument. But $R(F)$ is not upper-semicontinuous in the L^p weak topology. Second, $R(F)$ is invariant under the conformal group of dilations, rotations and translations, namely,

(2.3) $$F(x) \to F(\gamma \mathscr{R} x + y), \qquad \gamma > 0, \quad \mathscr{R} \in O(n), \quad y \in \mathbf{R}^n.$$

Furthermore, if $q = p'$, a case for which we shall explicitly find f, an inversion symmetry also exists, i.e.,

(2.4) $$F(x) \to |x|^{\lambda - 2n} F(x|x|^{-2}).$$

The existence of this large invariance group means that a maximizing f cannot be unique and also that it is easy for a weakly convergent maximizing sequence $\{f_n\}$ to converge to zero.

Third, if the kernel $|x - y|^{-\lambda}$ is changed slightly, a maximizing f need not exist. Explicitly, let $K(r)$, for $r > 0$, be any positive function such that $r^\lambda K(r)$ is strictly monotone increasing. Consider $K(|x - y|)$ in place of $|x - y|^{-\lambda}$ as a kernel in (2.1). The Fourier transform of the Bessel potential, $(1 + |x - y|^2)^{-\lambda/2}$, is a good example; it is even positive definite. For any $f \in L^p$, let $F(x) = |f(x/2)|$. It is easy to check that $R(F) > R(f)$, and hence, that no maximum can exist.

One of the key tools we shall use (several times, in fact) is Riesz's rearrangement inequality [24] (for a generalization see [7]) in the strong form given by Lieb [20]. It is recalled in Lemma 2.1.

Definition 1. Let $f: \mathbf{R}^n \to \mathbf{C}$ satisfy $\mu_f(a) = \mu\{x \mid |f(x)| > a\} < \infty$ for all $a > 0$. (Here, μ is Lebesgue measure.) $f^*: \mathbf{R}^n \to [0, \infty)$ is a *symmetric decreasing rearrangement* of f if $f^*(x)$ depends only on $|x|$, and $f^*(x_1) \geq f^*(x_2) \geq 0$ if $|x_1| \leq |x_2|$, and $\mu_{f^*}(a) = \mu_f(a)$, for all $a > 0$.

It is easy to check that f^* always exists and it is defined uniquely almost everywhere (see [20]). Henceforth, notation will be abused in the sense that any function $f(x)$ that depends only on $|x|$ will sometimes be written as $f(|x|)$.

It is convenient to introduce the following sets of functions from $\mathbf{R}^n \to [0, \infty)$ (where T denotes "translate"):

SD $= \{f \mid f$ is symmetric decreasing, i.e., $f = f^*\}$;
SSD $= \{f \mid f \in$ SD and f is strictly monotone decreasing$\}$;
TSD $= \{f \mid f_y(x) = f(x + y)$ and $f_y \in$ SD for some $y \in \mathbf{R}^n\}$;
TSSD $= \{f \mid f \in$ TSD and $f_y \in$ SSD$\}$.

LEMMA 2.1. *Let f, g, h be functions on $\mathbf{R}^n$ satisfying the conditions of Definition* 1 *and let*

$$I(f, g, h) = \int\int f(x)g(x - y)h(y)\, dx\, dy.$$

Then (i) $I(f^*, g^*, h^*) \geq |I(f, g, h)|$.

If, in addition, $g^* \in$ SSD *then*

(ii) $I(f^*, g^*, h^*) > |I(f, g, h)|$ *unless* $f(x) = f^*(x + y)$ *and* $h(x) = h^*(x + y)$ *for some (common)* $y \in \mathbf{R}^n$.

The first part of this lemma has been generalized to more than three functions and more than two variables in [7].

Another closely related fact that will be needed later is Lemma 2.2. We omit the easy proof (which mimics the proof of Lemma 2.1); it can also be obtained from Lemma 2.1 by suitable choice of h.

LEMMA 2.2. *Let $g = g^*, f = f^*$. Suppose the convolution $k \equiv g * f$ satisfies $k(x) < \infty$ for all $x \neq 0$. Then*

(i) $k = k^*$ *and*
(ii) $k \in$ SSD *if* $g \in$ SSD.

The existence theorem to be proved is

THEOREM 2.3. *Let $1/p + \lambda/n = 1 + 1/q$ with $0 < \lambda < n$, $1 < p, q < \infty$. Then*

(i) $N_{p,\lambda,n}$ *in* (2.2) *is finite and there exists an $f \in L^p$ that maximizes R, i.e.,* $R(f) = N_{p,\lambda,n}$.

(ii) *After multiplication by a suitable complex constant, every maximizing f is in* TSSD *and satisfies the pair of equations*

$$(2.5)\qquad |x|^{-\lambda} * f = g^{t-1}, \qquad |x|^{-\lambda} * g = f^{p-1}$$

for some $g \in L^t$ *and* $g \in$ TSSD. (*Here* $t = q' = q/(q-1)$.) *After a common translation*, $f, g \in$ SSD.

(iii) *When* $q' = p = t$, *then* $g = f$.

(iv) *Let* $q' = p = t$ *and let f be translated so that* $f = f^*$. *Then, possibly after a dilation* $f(r) \to \gamma^{n/p} f(\gamma r)$, *f has the inversion symmetry of* (2.4):

$$(2.6)\qquad f(1/r) = r^{2n/p} f(r).$$

In the following, irrelevant positive constants will all be denoted by the common symbol C.

Proof of (ii) *and* (iii): These two parts are easy in view of Lemma 2.1. N in (2.2), which is here assumed to be finite, can be written as

$$(2.7)\qquad N_{p,\lambda,n} = \sup_{f,g} \int\int f(x) g(y) K(x-y)\, dx\, dy / \|f\|_p \|g\|_t,$$

where $f \in L^p$, $g \in L^t$ and $K(x) = |x|^{-\lambda}$ and $1/t + 1/p + \lambda/n = 2$. Since the rearrangements $f \to f^*$, $g \to g^*$ do not change the norms $\|f\|_p$, $\|g\|_t$, and since $K = K^* \in$ SSD, Lemma 2.1 (ii) shows that $f, g \in$ TSD (possibly after the multiplication by constants). The equations (2.3) are then easy to derive by letting $f \to f + \varepsilon\varphi$, $g \to g + \delta\psi$ and setting the derivatives at $\varepsilon = \delta = 0$ equal to zero. (Again, it may be necessary to multiply f and g by constants to get unity on the right side of (2.5).) By Lemma 2.2, equations (2.5) imply that (after a translation), $f, g \in$ SSD. (iii) follows from (2.7) and the fact that $K(x-y)$ is positive definite. In fact, $|x|^{-\lambda} = C|x|^{-(n+\lambda)/2} * |x|^{-(n+\lambda)/2}$. (See (3.6).) □

Note that (2.7) implies

$$(2.8)\qquad N_{p,\lambda,n} = N_{t,\lambda,n}, \qquad 1/p + 1/t + \lambda/n = 2.$$

Beginning of Proof of (i). Let $\{f_j\}$ be a maximizing sequence, i.e., $R(f_j) \to N$. Assume $\|f_j\|_p = 1$. Since $f_j \in L^p$, f_j^* exists and $\|f_j^*\|_p = 1$. By Lemma 2.1 (see the proof of (ii)), $R(f_j^*) \ge R(f_j)$, so we can henceforth assume that $f_j = f_j^*$. Now

$$(2.9)\qquad 1 = C\int_0^\infty r^{n-1} f_j(r)^p\, dr \ge C\int_0^R r^{n-1} f_j(r)^p\, dr \ge C R^n f_j(R)^p$$
$$\Rightarrow 0 \le f_j(r) \le C r^{-n/p}.$$

By passing to a subsequence, we can then assume

$$(2.10)\qquad f_j(r) \to f(r) \quad \text{and} \quad 0 \le f(r) \le C r^{-n/p}$$

for all rational r. Since $f_j(r)$ is non-increasing in r, it is easy to see that $f_j(r)$ converges for almost all $r > 0$ and therefore that $f = f^*$. (This is essentially Helly's theorem.) By Fatou's lemma, $f \in L^p$.

The problem we face is that f could easily be zero because of the dilation subgroup mentioned in (2.3). Even if $f \neq 0$ it is not obvious that $R(f) \geq N_{p,\lambda,n}$, but this fact will be proved with the help of the following lemma.

LEMMA 2.4. *Let* $1/p + \lambda/n = 1 + 1/q$, *with* $0 < \lambda < n$. *Suppose* $f \in L^p(\mathbf{R}^n)$ *is spherically symmetric and* $|f(r)| \leq \varepsilon r^{-n/p}$ *for all* $r > 0$. *There is a constant,* C_n, *independent of* f *and* ε *such that* $\||x|^{-\lambda} * f\|_q \leq C_n \|f\|_p^{p/q} \varepsilon^{1-p/q}$. (*Note that* $p < q$.)

Remarks. (i) Lemma 2.4 and (2.9) obviously imply that $N < \infty$.

(ii) Lemma 2.4 follows from known results about the Lorentz spaces $L(p, q)$ (see [22], [9], [28]). We give our own proof, which is based on a transformation to *logarithmic radial variables*, for two reasons: (a) In conjunction with Lemma 2.1 it provides an alternative strategy for proving many known facts about $L(p, q)$ spaces; (b) The formulation given in our proof will be needed later in order to establish (and hence to exploit) the inversion symmetry for $q = p'$ given in Theorem 2.3 (iv).

Proof. Define F: $\mathbf{R} \to \mathbf{R}$ by

$$F(u) = e^{un/p} f(e^u), \quad \text{whence} \tag{2.11}$$

$$\sigma_n^{1/p} \|F\|_{L^p(\mathbf{R})} = \|f\|_{L^p(\mathbf{R}^n)}. \tag{2.12}$$

Here, σ_n is the area of the unit sphere in $\mathbf{R}^n$,

$$\sigma_n = 2\pi^{n/2}/\Gamma(n/2). \tag{2.13}$$

Without loss of generality, we can assume $f(r) \geq 0$. Define $h = |x|^{-\lambda} * f$, which is spherically symmetric, and H: $\mathbf{R} \to \mathbf{R}$ by $H(u) = e^{un/q} h(e^u)$. As in (2.11), $\sigma_n^{1/q} \|H\|_q = \|h\|_q$. An explicit form for H, which is easily obtained by integrating $d^n x$ over angles in $\mathbf{R}^n$, is the following:

$$H(u) = \int_{-\infty}^{\infty} L_n(u - v) F(v)\, dv, \quad \text{where} \tag{2.14}$$

$$L_n(u) = 2^{-\lambda/2} \exp\{u(n/q - \lambda/2)\} Z_n(u), \tag{2.15}$$

$$Z_n(u) = \sigma_{n-1} \int_0^{\pi} [\cosh u - \cos\theta]^{-\lambda/2} (\sin\theta)^{n-2}\, d\theta, \qquad n \geq 2, \tag{2.16}$$

$$= [\cosh u + 1]^{-\lambda/2} + [\cosh u - 1]^{-\lambda/2}, \qquad n = 1.$$

Now, $L_n \in L^1(\mathbf{R})$ and $F \in L^p(\mathbf{R})$ and $\|F\|_\infty < \varepsilon$. (Note that $|n/q - \lambda/2| < \lambda/2$, since $p > 1$, and that the singularity, if any, in $Z_n(u)$ for u near zero is no worse than $|u|^{-\lambda/n}$.) By Young's inequality, $\|H\|_p \leq C\|F\|_p$ and $\|H\|_\infty \leq C\|F\|_\infty \leq C\varepsilon$. Since $q > p$, the lemma follows from Hölder's inequality. □

Before returning to the proof of Theorem 2.3(i), let us draw two conclusions from the construction, (2.11)–(2.16).

First, the original problem (2.2) is equivalent to the one-dimensional problem

$$(2.17)\qquad N_{p,\lambda,n} = \sigma^{1/q-1/p}\sup\left\{ \|L_n * F\|_q/\|F\|_p \,\middle|\, 0 \neq F \in L^p(\mathbf{R})\right\}.$$

In particular, $L_n *$ is a bounded operator from $L^p(\mathbf{R})$ to $L^q(\mathbf{R})$.

The second conclusion is

Proof of Theorem 2.3 (iv). Make the change of variables given in (2.10) and note that $n/q - \lambda/2 = 0$ in (2.14) when $q = p'$. Thus, $L_n = K_n \in \mathrm{SSD}(\mathbf{R})$. From (2.17) and by the same proof as for Theorem 2.3(ii), $F \in \mathrm{TSSD}(\mathbf{R})$. Translating F, namely $F(u) \to F(u + y)$, is the same as dilation of f. With $F \in \mathrm{SSD}(\mathbf{R})$, inverting (2.11) gives the desired result. □

It is worth noting that the strong rearrangement inequality had to be used *twice* to prove Theorem 2.3(iv).

Lemma 2.4 and (2.9) not only imply that $N < \infty$, they also imply that, after a suitable j-dependent dilation, we can assume that *the limit*, f, *in* (2.9) *is not zero*. To see this, let

$$a_j = \sup_r r^{n/p} f_j(r).$$

By Lemma 2.4, $a_j \nrightarrow 0$, for otherwise $\||x|^{-\lambda} * f_j\|_q \to 0$ while $\|f_j\|_p = 1$, which would mean that $\{f_j\}$ is not a maximizing sequence. Thus, $a_j > 2\beta > 0$. Replace $f_j(r)$ by $\gamma_j^{n/p} f_j(\gamma_j r)$, which does not change the norm of f_j. We can now choose γ_j so that $f_j(1) > a_j/2 > \beta > 0$. Therefore, $f(1) > \beta$ and, since $f \in \mathrm{SD}$, $f(r) > \beta$ for $r \leq 1$. Thus, f is not zero.

Let us briefly review the situation about Theorem 2.3(i). We have a maximizing sequence $\{f_j\}$ of non-negative symmetric decreasing functions which converge pointwise, almost everywhere to $f \neq 0$. By Fatou's lemma, $\|f\|_p \leq \lim\|f_j\|_p = 1$; therefore f will be maximizing if $I(f_j) \to I(f)$, where

$$(2.18)\qquad I(g) \equiv \||x|^{-\lambda} * g\|_q.$$

The convergence of $I(f_j)$ to $I(f)$ will be proved, but only after we first prove that $R(f_j) \to R(f)$ and $\|f\|_p = 1$. Before doing so let us first consider a related problem which is interesting in its own right, for which it is easy to establish that $I(f_j) \to I(f)$. This other problem and its solution are stated as the following theorem.

THEOREM 2.5. *Let $1/p + 1/t + \lambda/n = 2$ with $0 < \lambda < n$ and $1 < p$, $t < \infty$ as before, and consider the ratio in* (2.6) *but with g restricted to be f; i.e.,*

$$\tilde{N} = \sup_f \int\int f(x)f(y)|x-y|^{-\lambda}\,dx\,dy/\|f\|_p\|f\|_t \tag{2.19}$$

with $f \in L^p \cap L^t$ and $f \neq 0$. (Naturally, $\tilde{N} \leq N$ and $\tilde{N} = N$ when $t = p = q'$ as stated in Theorem 2.3(iii).) *Let $t \neq p$. Then there exists a maximizing f for $\tilde{N}$. Furthermore, after multiplication by a constant, a dilation and a translation (i.e., $f(x) \to cf(\gamma x + y)$ this f is in* SSD *and satisfies*

$$|x|^{-\lambda} * f = f^{p-1} + f^{t-1}. \tag{2.20}$$

Proof. All of the argument is as before, but with one additional fact at our disposal. We can (after dilation and multiplication by a constant) assume that $\|f_j\|_p = \|f_j\|_t = 1$. From (2.9) the limit f satisfies $f(r) \leq Cr^{-n/p}$ and $f(r) \leq Cr^{-n/t}$ (same C). Let $h(x) = C\min\{|x|^{-n/p}, |x|^{-n/t}\}$. Although h is neither in L^p nor in L^t, the function $h(x)h(y)|x-y|^{-\lambda} \in L^1(\mathbf{R}^n \times \mathbf{R}^n)$. (To see this, note that $h \in L^s$ when $\min(t,p) < s < \max(t,p)$. Choose s so that $1/s + 1/s + \lambda/n = 2$. But we already proved that $h \to |x|^{-\lambda} * h$ is bounded from L^s to $L^{s'}$.) Therefore, if $I(f)$ denotes the integral in (2.19), we have that $I(f_j) \to I(f)$ by dominated convergence. □

Returning to Theorem 2.3(i), we see that establishing the convergence of $I(f_j)$ to $I(f)$ is more delicate than in Theorem 2.5, even if $p \neq t$, because all we know is that $f \in L^p$, and not necessarily in L^t. Therefore, the dominated convergence argument cannot be used.

To control the convergence of $R(f_j)$ to $R(f)$, the following lemma due to Brezis and myself [8] is useful.

LEMMA 2.6. *Let $0 < p < \infty$. Let (M, Σ, μ) be a measure space and let $\{f_j\}$ be a uniformly norm-bounded sequence in $L^p(M, \Sigma, \mu)$ that converges pointwise, almost everywhere to f. (By Fatou's lemma $f \in L^p$.) Then the following limit exists and equality holds.*

$$\lim_{j\to\infty} \int \left| |f_j(x)|^p - |f(x) - f_j(x)|^p - |f(x)|^p \right| d\mu(x) = 0.$$

Remarks. Lemma 2.6 says more than that $\|f_j\|_p^p - \|f - f_j\|_p^p \to \|f\|_p^p$. It improves Fatou's lemma which says that $\liminf \|f_j\|_p^p \geq \|f\|_p^p$. In [8] a similar theorem is proved for functionals of the form $f \to \int J(f)\,d\mu$. The conclusion of Lemma 2.6 does not hold (except when $p = 2$) if pointwise is replaced by weak convergence. Note that the lemma holds even for $0 < p < 1$. Lemma 2.6 can be proved simply without using the general results in [8]. Note that

$$\left||f_j|^p - |f - f_j|^p - |f|^p\right| \le \begin{cases} 2|f|^p, & 0 < p \le 1 \\ p2^{p-1}\{|f - f_j|^{p-1}|f| + |f - f_j||f|^{p-1}\}, & 1 \le p < \infty. \end{cases}$$

The lemma follows from the first inequality for $0 < p < 1$ by dominated convergence; for $1 \le p < \infty$ it follows from the second by Egorov's theorem.

The utility of Lemma 2.6 for problems in the calculus of variations is given in the next lemma.

LEMMA 2.7. *Let* (M, Σ, μ) *and* (M', Σ', μ') *be measure spaces and let* X *(resp.* Y*) be* $L^p(M, \Sigma, \mu)$ *(resp.* $L^q(M', \Sigma', \mu')$*) with* $1 \le p \le q < \infty$. *Let* A *be a bounded linear operator from* X *to* Y. *For* $f \in X$, $f \ne 0$ *let*

$$R(f) = \|Af\|_Y/\|f\|_X \quad \text{and} \quad N = \sup\{R(f) | f \ne 0\}.$$

Let $\{f_j\}$ *be a uniformly norm-bounded maximizing sequence for* N *and suppose that* $f_j \to f \ne 0$ *and that* $Af_j \to Af$ *pointwise almost everywhere. Then* f *maximizes, i.e.,* $R(f) = N$. *Moreover, if* $p < q$ *and if* $\lim\|f_j\|_X = C$ *exists, then* $\|f\|_X = C$, *and hence* $\|Af_j\|_Y \to \|Af\|_Y$.

Proof. By Lemma 2.6, $\|f_j\|_X^p = \|f - f_j\|_X^p + \|f\|_X^p + o(1)^p$ (where $o(1)$ denotes something that goes to zero as $j \to \infty$) and $\|Af_j\|_Y^q = \|Af - Af_j\|_Y^q + \|Af\|_Y^q + \tilde{o}(1)^q$. If $a, b, c \ge 0$, then $(a^q + b^q + c^q)^{p/q} \le a^p + b^p + c^p$. Thus,

$$R(f_j)^p \le \left\{\|Af\|_Y^p + \|A(f - f_j)\|_Y^p + o(1)^p\right\}/\left\{\|f\|_X^p + \|f - f_j\|_X^p + \tilde{o}(1)^p\right\}.$$

Now $\|A(f - f_j)\|_Y \le N\|f - f_j\|_X$ for every j, and $o(1), \tilde{o}(1) \to 0$, and $R(f_j)^p \to N^p$. Since $f \ne 0$, we must have that $\|Af\|_Y \ge N\|f\|_X$, and hence $\|Af\|_Y = N\|f\|_X$. We also have that $\lim\|A(f - f_j)\|_Y - N\|f - f_j\|_X = 0$. For the last part, let $p < q$ and $\lim\|f_j\|_X = C$. Then $0 < \|f\|_X \le C$. Suppose $0 < \|f\|_X = D < C$. Then $\lim\|f - f_j\|_X^p = C^p - D^p > 0$. However, $(a^q + b^q)^{p/q} < a^p + b^p$ unless $a = 0$ or $b = 0$. Thus, $\lim R(f_j)^p < N^p$, which is a contradiction. □

The last sentence of Lemma 2.7, and the proof, were pointed out to me by Haïm Brezis.

Conclusion of Proof of Theorem 2.3(i). Let us return to the one-dimensional equivalent formulation given in (2.11)–(2.17). Given the maximizing sequence $f_j \in \mathrm{SD}$ with $\|f_j\|_p = 1$, (2.11) defines a sequence F_j: $\mathbf{R} \to \mathbf{R}$ with $\|F_j\|_p = \sigma_n^{-1/p}$ and $\|F_j\|_\infty \le C$ by (2.9). Also, $f_j \to f \ne 0$ pointwise, so $F_j \to F \ne 0$ pointwise.

Lemma 2.7 can now be applied to finish the proof provided the operator $A = L_n *: L^p(\mathbf{R}) \to L^q(\mathbf{R})$ satisfies $AF_j \to AF$ pointwise. But since $L_n \in L^1(\mathbf{R})$ and since $\|F_j - F\|_\infty < C$, $L_n * (F_j - F)(x) \to 0$ everywhere by dominated convergence. The fact that $I(f_j) \to I(f)$ is contained in the last sentence of Lemma 2.7. In our case, $p < q$ since $\lambda/n < 1$. □

III. The maximizing function when $p' = q$ or $p = 2$ or $q = 2$

In certain cases the equations (2.5) for the maximizing f can be solved and the constant N computed explicitly. In these cases a solution to (2.5) can be easily guessed and verified. The difficult part will be to prove that this f is actually maximizing. To prove this it will be necessary to use stereographic projection to recast (2.5) as an equation on the sphere $\mathbf{S}^n$.

Recall that $1/p + 1/t + \lambda/n = 2$ or $1/p + \lambda/n = 1 + 1/q$ with $t = q' = q/(q-1)$ and $0 < \lambda < n$, $1 < t, p, q < \infty$.

THEOREM 3.1. *When* $t = p = 2n/(2n - \lambda)$, $q = 2n/\lambda$ (*or* $p' = q$), *the maximizing* f *for* (2.1) *is* (*after multiplication by a constant and dilation*) *uniquely*

$$f(x) = (1 + |x|^2)^{-n/p} \quad \text{and} \tag{3.1}$$

$$N_{p,\lambda,n} = \pi^{\lambda/2} \frac{\Gamma(n/2 - \lambda/2)}{\Gamma(n - \lambda/2)} \left\{ \frac{\Gamma(n/2)}{\Gamma(n)} \right\}^{-1+\lambda/n}. \tag{3.2}$$

COROLLARY 3.2. (i) *Let* $q = t = 2$ *and* $p = 2n/(3n - 2\lambda)$, *which requires* $n < 2\lambda < 2n$. *The maximizing* f *for* (2.1) *is* (*after multiplication by a constant and dilation*) *uniquely*

$$f(x) = (1 + |x|^2)^{-n/p} \quad \text{and} \tag{3.3}$$

$$N_{p,\lambda,n} = \pi^{\lambda/2} \frac{\Gamma(n/2 - \lambda/2)}{\Gamma(\lambda/2)} \left\{ \frac{\Gamma(\lambda - n/2)}{\Gamma(3n/2 - \lambda)} \right\}^{1/2} \left\{ \frac{\Gamma(n/2)}{\Gamma(n)} \right\}^{-1+\lambda/n}. \tag{3.4}$$

(ii) *Let* $p = 2$, $t = 2n/(3n - 2\lambda)$, $q = 2n/(2\lambda - n)$, *which requires* $n < 2\lambda < 2n$. *The maximizing* f *for* (2.1) *is* (*after multiplication by a constant and dilation*) *uniquely*

$$f = |x|^{-\lambda} * (1 + |x|^2)^{-n/t} \tag{3.5}$$

and $N_{2,\lambda,n}$ *is given by the right side of* (3.4).

Proof of Corollary 3.2. Note that for $0 < \lambda, \gamma < n$ and $\lambda + \gamma > n$,

(3.6) $$|x|^{-\lambda} * |x|^{-\gamma} = D(\lambda, \gamma)|x|^{n-\lambda-\gamma},$$

(3.7) $$D(\lambda, \gamma) = \pi^{n/2}\Gamma(n/2 - \lambda/2)\Gamma(n/2 - \gamma/2)\Gamma(\lambda/2 + \gamma/2 - n/2) \times \{\Gamma(\lambda/2)\Gamma(\gamma/2)\Gamma(n - \lambda/2 - \gamma/2)\}^{-1}.$$

This follows from the fact [28, Theorem IV. 4.1] that the Fourier transform $w_\lambda(k) = \int |x|^{-\lambda}\exp(ik \cdot x)\,dx$ is

(3.8) $$w_\lambda(k) = |k|^{\lambda - n}\pi^{n/2}2^{n-\lambda}\Gamma(n/2 - \lambda/2)/\Gamma(\lambda/2).$$

When $q = 2$,

$$N^2 = \| |x|^{-\lambda} * f\|_2^2/\|f\|_p^2 = D(\lambda, \lambda)(f, |x|^{n-2\lambda} * f)/\|f\|_p^2.$$

But this maximization problem is, by Theorem 2.3(iii) and (2.7), the same as in Theorem 3.1 (but with $\lambda \to 2\lambda - n$). When $p = 2$ the proof is similar, using (2.5). See (2.8). □

Beginning of Proof of Theorem 3.1. By Theorem 2.3, the f we seek *must have* (after dilation, etc.) two properties:

(a) The inversion symmetry $f(1/r) = r^{2n/p}f(r)$ and

(b) It satisfies (2.5) up to a constant, namely

(3.9) $$|x|^{-\lambda} * f = Bf^{p-1}.$$

Clearly, (a) holds for (3.1). The fact that (3.1) satisfies (3.9) can be seen in several ways. One way is to note that

(3.10) $$f_\mu(x) \equiv (1 + |x|^2)^{-\mu}$$

has the Fourier transform

(3.11) $$\hat{f}_\mu(k) = (2\pi)^{n/2}\int_0^\infty (|k|r)^{1-n/2}r^{n-1}J_{-1+n/2}(|k|r)f_\mu(r)\,dr = \pi^{n/2}2^{1-\mu+2/n}\Gamma(\mu)^{-1}|k|^{\mu-n/2}K_{\mu-n/2}(|k|).$$

Here, K is a Bessel function and satisfies $K_\nu(x) = K_{-\nu}(x)$. If we set $\mu = n/p = n - \lambda/2$ and use (3.8), we find that

(3.12) $$|x|^{-\lambda} * f_{n/p} = B_\lambda f_{\lambda/2} = B_\lambda (f_{n/p})^{p-1},$$

(3.13) $$B_\lambda = \pi^{n/2}\Gamma(n/2 - \lambda/2)/\Gamma(n - \lambda/2).$$

It follows that $R(f_{n/p})$ in (2.1) is

(3.14) $$\| |x|^{-\lambda} * f_{n/p}\|_q/\|f_{n/p}\|_p = B_\lambda[\hat{f}_n(0)]^{1/q-1/p} = \text{right side of (3.2)}.$$

The calculation just given is slightly formal, but it can easily be made rigorous. The real problem that faces us, however, is this: (3.12) shows that f in (3.1) (hereafter we shall denote $f_{n/p}$ by f) satisfies (3.9). It also has the correct inversion symmetry. Is this f maximizing? Is it the unique maximizer (up to a constant)? We do not know that (3.9) has an (essentially) unique solution—even if we restrict to the SSD category—and we shall offer no proof of this kind of uniqueness. *This is an open problem*! But it will be shown that f is (essentially) unique in the category of maximizers. In the course of this proof, (3.12)–(3.14) will be rederived in a simpler way. For the proof, a change of variables will be required, namely stereographic projection.

Stereographic Projection. Consider the sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$, $\mathbf{S}^n = \{\Omega \in \mathbf{R}^{n+1} \mid |\Omega| = 1\}$. Consider the invertible map Σ: $\mathbf{R}^n \to \mathbf{S}^n \setminus (0,\ldots,0,-1)$

$$(3.15) \qquad \Sigma(x) = (\rho, \xi) = \left(2x/(1+|x|^2), (1-|x|^2)/(1+|x|^2)\right)$$

where $\rho \in \mathbf{R}^n$. Conversely, if $\rho \in \mathbf{R}^n$, $\xi \in (-1, 1]$ and $(\rho, \xi) \in \mathbf{S}^n$,

$$(3.16) \qquad \Sigma^{-1}((\rho, \xi)) = \rho/(1+\xi).$$

Apart from trivial constants this is the usual stereographic projection with $0 \in \mathbf{R}^n \to$ "north pole".

Let $x_1, x_2 \in \mathbf{R}^n$ and $\Omega_i = \Sigma(x_i)$ with $\Omega_i = (\rho_i, \xi_i)$. Then

$$(3.17) \qquad |x_1 - x_2| = |\Omega_1 - \Omega_2|\{(1+\xi_1)(1+\xi_2)\}^{-1/2}.$$

Here, $|\Omega_1 - \Omega_2|$ means Euclidean distance in $\mathbf{R}^{n+1}$, not geodesic distance on $\mathbf{S}^n$.

Let $d\Omega$ be the rotation invariant measure on $\mathbf{S}^n$ with the normalization

$$(3.18) \qquad \int d\Omega = \sigma_{n+1} = 2\pi^{(n+1)/2}\Gamma(n+1)/2)^{-1}$$

which is the area of the unit sphere in $\mathbf{R}^{n+1}$ (see (2.13)). Then the Jacobian of Σ is given by

$$(3.19) \qquad d\Omega = d\rho/|\xi| = 2^n(1+|x|^2)^{-n}\, dx = (1+\xi)^n\, dx.$$

With any f: $\mathbf{R}^n \to \mathbf{C}$ we associate F: $\mathbf{S}^n \to \mathbf{C}$ (denoted by $f \leftrightarrow F$) by

$$(3.20) \qquad F(\Omega) = (1+\xi)^{-\mu} f(\Sigma^{-1}(\Omega)),$$

$$f(x) = 2^{-n}(1+|x|^2)^{\mu} F(\Sigma(x)),$$

with $\mu = n/p = n - \lambda/2$. ($\lambda$ enters at this point.) Clearly,

$$(3.21) \qquad \|F\|_p = \|f\|_p.$$

In particular, f given by (3.1) corresponds to $F(\Omega) = \text{constant} = 2^{-\mu}$.

From (3.15)–(3.18) we have that when $f \leftrightarrow F$,

$$\{|x|^{-\lambda} * F\}(x) \leftrightarrow (1+\xi)^{\lambda-n}\{|\Omega|^{-\lambda} * F\}(\Omega), \quad \text{where} \tag{3.22}$$

$$(|\Omega|^{-\lambda} * F)(\Omega) = \int d\Omega' |\Omega - \Omega'|^{-\lambda} F(\Omega'). \tag{3.23}$$

(Again, note that Euclidean distance in $\mathbf{R}^{n+1}$ is used.) Equation (3.9) then takes the following simple form (with the same B):

$$|\Omega|^{-\lambda} * F = BF^{p-1}. \tag{3.24}$$

This, together with (3.20), (3.21), gives another equivalent form for N (cf. (2.17)):

$$N_{p,\lambda,n} = \sup\left\{ \|\Omega^{-\lambda} * F\|_q / \|F\|_p \,|\, F \in L^p(S^n), F \neq 0 \right\}. \tag{3.25}$$

As stated before, $F = 2^{-\mu} \leftrightarrow f$ in (3.1). To check (3.12)–(3.14) we must compute

$$\begin{aligned} I &= \int d\Omega' |\Omega - \Omega'|^{-\lambda} \\ &= \sigma_n \int_0^{\pi} d\theta (\sin\theta)^{n-1} (2 - 2\cos\theta)^{-\lambda/2} \\ &= \sigma_n 2^{n-\lambda} \int_0^{\pi/2} d\varphi (\cos\varphi)^{n-1} (\sin\varphi)^{n-1-\lambda} \\ &= \sigma_n 2^{n-\lambda-1} \Gamma(n/2)\Gamma(n/2 - \lambda/2)/\Gamma(n - \lambda/2). \end{aligned} \tag{3.26}$$

Thus, $B = I 2^{\lambda-n} = B_\lambda$ in (3.13). Furthermore,

$$\begin{aligned} \| |\Omega|^{-\lambda} * F\|_q / \|F\|_p &= BF^{p-2} \sigma_{n+1}^{1/q - 1/p} \\ &= \text{right side of (3.2)}, \end{aligned} \tag{3.27}$$

if we use the duplication formula for the gamma function.

Equation (3.24) has one very great advantage over (3.9). The $O(n+1)$ rotation invariance of (3.24) allows us to generate new solutions from old ones. This fact will eventually permit us to conclude that (3.1) is the (essentially) unique maximizer.

As an interesting aside before returning to the proof of Theorem 3.1, let us consider some other solutions to (3.9) and (3.24). (Irrelevant constants will be suppressed here.)

(a) We have $f(x) = (1 + |x|^2)^{-\mu} \leftrightarrow F(\Omega) = 1$. However, by translation and dilation of f in $\mathbf{R}^n$, there is an $n+1$ parameter family of solutions as follows:

$$f(x) = \left[b^2 + |x - z|^2\right]^{-\mu} \leftrightarrow F(\Omega) = \left[1 + (w, \Omega)\right]^{-\mu}, \tag{3.28}$$

where $b \in \mathbf{R}$, $z \in \mathbf{R}^n$, $w \in \mathbf{R}^{n+1}$. All b, z and w are allowed, except for the condition $|w| < 1$. The SSD category corresponds to $z = 0$ and $w = (0, \ldots, 0, c)$ with $|c| < 1$.

These other solutions are interesting for the following reason. Since $F =$ const. satisfies (3.24) and since this solution has the maximum possible $O(n+1)$ symmetry, it might have been supposed that $F =$ const. is the unique maximizer. But we see from (3.28) that there are other equivalent solutions with less symmetry. This is indeed surprising.

(b) $f(x) = |x - z|^{-\mu}$ with $z \in \mathbf{R}^n$ also satisfies (3.9). It is not allowed since it is not in L^p. This is an n parameter family and the correspondence is

$$f(x) = |x - z|^{-\mu} \leftrightarrow F(\Omega) = (1 + \xi)^{-\mu/2}|\Omega - \Omega'|^{-\mu} \tag{3.29}$$

with $\Omega' = \Sigma(z)$. Even more solutions can be obtained from (3.29) by applying an $O(n+1)$ rotation to Ω and Ω'. The function $2(1+\xi)$ becomes $|\Omega - \Omega''|^2$, with $\Omega \neq \Omega'$. Thus we have a $2n$ parameter family of solutions:

$$F(\Omega) = |\Omega - \Omega'|^{-\mu}|\Omega - \Omega''|^{-\mu} \leftrightarrow f(x) = |x - z'|^{-\mu}|x - z''|^{-\mu} \tag{3.30}$$

with $\Omega', \Omega'', z', z''$ arbitrary except that $\Omega' \neq \Omega''$ and $z' \neq z''$. This is amusing because the rotation invariance of (3.24) allowed us to generate a nontrivial $2n$ parameter family of solutions starting from $|x|^{-\mu}$.

Conclusion of Proof of Theorem 3.1. The f given in (3.1) satisfies (3.9) and we want to show that it maximizes and that it is (essentially) unique.

Let f be any maximizer. By Theorem 2.3 we can assume three things (after translation and dilation): (a) f satisfies (3.9). This means, in particular, that $f(x)$ is defined for *all* x by the left side of (3.9), not for just almost every x; (b) $f \in$ SSD; (c) $f(1/r) = r^{2\mu}f(r)$. Let $F \leftrightarrow f$. By (b), $F(\Omega)$ depends only on ξ: $F(\Omega) = \varphi(\xi)$. By (c), $\varphi(\xi) = \varphi(-\xi)$. Thus, $f(x) = (1 + |x|^2)^{-\mu}\varphi((1 - |x|^2)/(1 + |x|^2))$. Let $R \in O(n+1)$ be the following rotation:

$$R\colon (\rho_1, \ldots, \rho_n, \xi) \to (\rho_1 \cos\theta - \xi \sin\theta, \rho_2, \ldots, \rho_n, \xi\cos\theta + \rho_1 \sin\theta).$$

By the rotation invariance of (3.24),

$$\begin{aligned} f_R(x) &= (1 + |x|^2)^{-\mu}F(R\Sigma(x)) \\ &= (1 + |x|^2)^{-\mu}\varphi([(1 - |x|^2)\cos\theta + 2x_1 \sin\theta]/[1 + |x|^2]) \end{aligned}$$

also maximizes. Therefore, by Theorem 2.3, there exists a unique $y \in \mathbf{R}^n$ such that $f_R(x + y) \in$ SSD. This y is the ***unique*** solution to $f_R(y) = \max_x f_R(x)$. By the $O(n-1)$ rotation invariance of f_R in $(x_2, \ldots, x_n)$ we see that $y_2 = \cdots = y_n = 0$. Since $f_R(x + y) \in \mathrm{SSD}(\mathbf{R}^n)$, $g(x_1) \equiv f_R(x_1 + y_1, x_2, \ldots, x_n) \in \mathrm{SSD}(\mathbf{R}^1)$ for any fixed $x_2, \ldots, x_n$. But $f_R((1, 1, \ldots, 1)) = f_R((-1, 1, \ldots, 1))$. Therefore $y_1 = 0$ also, and $y = 0$. This means that $\varphi([\ ]/[\ ])$ is spherically symmetric in x for *all* θ and I claim that φ must then be a constant. To see this, let $u_+, u_- \in [-1, 1]$ and let $x_{\pm} = (\pm b, 0, \ldots, 0)$. Since $f_R(x_+) = f_R(x_-)$, we shall have $\varphi(u_+) = \varphi(u_-)$ if we can find b and θ such that $u_{\pm} = [(1 - b^2)\cos\theta \pm 2b\sin\theta]/[1 + b^2]$. Let $b = -\tan(\psi/2)$. These equations then read $u_{\pm} = \cos(\psi \pm \theta)$, and we see that a solution is trivial. □

It should be noted that the proof above used the strong rearrangement inequality (Lemma 2.1) and Lemma 2.2 *twice*: Once to show that any maximizer, f, is in TSSD and once to show that $f(1/r) = r^{2\mu} f(r)$. For the latter, the formulation in (2.17) was essential.

The following is another way to conclude the proof of Theorem 3.1. It uses the rearrangement inequality on $\mathbf{S}^n$ of Baernstein and Taylor [3] which is a generalization of the inequality on $\mathbf{S}^1$ of Friedberg and Luttinger [13]. The inequality is:

(3.31)
$$\int\int F(\Omega) K(\Omega \cdot \Omega') G(\Omega')\, d\Omega\, d\Omega' \leq \int\int F^*(\Omega) K(\Omega \cdot \Omega') G^*(\Omega')\, d\Omega\, d\Omega',$$

where $K\colon [-1, 1] \to \mathbf{R}$ is non-decreasing, and where F^* is equimeasurable with F, $F^*(\rho, \xi)$ depends only on ξ and is non-increasing in ξ (and likewise for G^*). Unfortunately, Baernstein and Taylor do not prove a strong inequality analogous to Lemma 2.1 (ii), which, it may be conjectured, exists. If it did hold, then the proof could be simplified.

In our case $K(\Omega \cdot \Omega') = |\Omega - \Omega'|^{-\lambda}$ which is strictly increasing. Let $F(\Omega) = \varphi(\xi)$ be the maximizer that satisfies $\varphi(\xi) = \varphi(-\xi)$. If the strong version of (3.31) held, we could immediately conclude that $\varphi(\xi)$ is either non-increasing or non-decreasing, in which case $\varphi(\xi) =$ constant and the proof would be finished. In the absence of this fact let $F^*(\Omega) = \psi(\xi)$ with $\psi(\xi)$ non-increasing. By (3.31), F^* also maximizes. Using (3.20), $F^* \leftrightarrow h$ with $h(x) = h(|x|)$. By the strong rearrangement inequality on $\mathbf{R}^n$, for some $\gamma > 0$, $h_\gamma(x) \equiv h(\gamma x)$ satisfies $h_\gamma(1/r) = r^{2\mu} h_\gamma(r)$. In general (without assuming any symmetry), if $F \leftrightarrow f$, then $F_\gamma \leftrightarrow f_\gamma$, where $F_\gamma(\rho, \xi) = F(2\lambda\rho/w, \nu/w)$ with $w = 1 + \xi + \gamma^2(1 - \xi)$ and $\nu = 1 + \xi - \gamma^2(1 - \xi)$. In our case $(F^*)_\gamma(\Omega) = \psi_\gamma(\xi) = \psi(\nu/w)$ and $\psi_\gamma(\xi) = \psi_\gamma(-\xi)$. Setting $\xi = 1$ we conclude that $\psi(1) = \psi(-1)$. Since ψ is non-increasing, $\psi =$ constant and the proof is completed.

IV. The Sobolev inequality

As another application of the method of Section II, we shall prove here the existence of a maximizing function f on $\mathbf{R}^n$ for the sharp constant in the Sobolev inequality

$$K_n \|\nabla f\|_2 \geq \|f\|_{2^*}, \qquad 2^* = 2n/(n-2), \quad n \geq 3. \tag{4.1}$$

I thank H. Brezis and P. L. Lions for suggesting to me that there should be a simple, direct "rearrangement inequality proof" of the existence of this f. Existence proofs already exist, of course (see [2], [31] and also [25]). (A generalization of (4.1) using Lorentz space norms was given by Alvino [1].) What is offered below, it is hoped, is a more direct and simpler argument.

A generalization of (4.1), useful in the theory of the Schroedinger equation, was given in [14] for $n = 3$:

$$K_{n,p}\|\nabla f\|_2 \geq \||x|^{-b}f\|_p, \qquad n \geq 3, \tag{4.2}$$

for $0 \leq b < 1$ and $p = 2n/(2b + n - 2)$.

In [14], an interesting extension of (4.2) is also given. If $1 - n/2 < b < 0$, no inequality of this type is possible for all f. But if f is restricted to be *spherically symmetric* (*not necessarily* symmetric decreasing), then a bound as in (4.2) holds and there is also a maximizing f. This is also given in Theorem 4.3.

Generalizations of (4.2) can be found in [12] and [21]. Flett [12] gives [21], Theorem 6.5.8, as the earliest reference to (4.2). Glaser, Martin, Grosse and Thirring [14] were unaware of this, but seem to have been the first to compute the sharp constant in (4.2). Generalizations of (4.1) can be found in [32], [33]. See also [5], [16].

It will be recalled that the rearrangement inequality, Lemma 2.1, was used *twice* in the proof of Theorem 2.3. First, it was used to show that a maximizing sequence could be sought in the SD($\mathbf{R}^n$) category. Second, when $p' = q$, it was used in the one-dimensional formulation (2.11)–(2.17) to deduce (2.6). The dual usage will also be needed here because we are faced with the same problem as that outlined in the beginning of Section II: The variational problem posed by (4.1) and (4.2) is invariant under the same conformal group (including inversion).

We begin with the fact that for $f \in W^{1,p}(\mathbf{R}^n) = \{f | f \text{ and } \nabla f \in L^p(\mathbf{R}^n)\}$, $\|\nabla f\|_p \geq \|\nabla f^*\|_p$ where f^* is the symmetric decreasing rearrangement of $|f|$. This fact has been known for a long time (see [10], [23], for example), but all proofs of it seem to be complicated. There is one case, suitable for our purposes here, in which the following simple proof can be given [20], and it would be desirable to be able to extend this argument to the $W^{1,p}$ case, $p \neq 2$. It would also be desirable to have a strict inequality as in Lemma 2.1(ii).

LEMMA 4.1. *Let* $f \in W^{1,2}(\mathbf{R}^n) = H^1(\mathbf{R}^n)$. *Then* $f^* \in H^1(\mathbf{R}^n)$ *and* $\|\nabla f\|_2 \geq \|\nabla f^*\|_2$.

Proof. Let $t > 0$ and $g_t(x - y) = e^{t\Delta}(x, y)$ be the kernel of the heat equation semigroup. Let $f \in L^2$. It is easy to see that if $A(f, t) \equiv (1/t)[\|f\|_2^2 - (f, g_t * f)]$, then $\lim_{t \to 0} A(f, t) = \|\nabla f\|_2^2$ or $+\infty$ according as $\nabla f \in L^2$ or not. (See [20] for a proof of this fact.) For each $t > 0$, $g_t(\cdot)$ is a

Gaussian and hence in SSD($\mathbf{R}^n$). Therefore, $(f, g_t * f) \le (f^*, g_t * f^*)$ by Lemma 2.1. Since $\|f\|_2 = \|f^*\|_2$, the lemma is proved. □

With this preparation we can now prove the following about $\mathbf{R}^1$:

THEOREM 4.2. *Let $F \in H^1(\mathbf{R})$ and $2 < p < \infty$. For $F \ne 0$ let*

$$T(F) = \|F\|_p^2/\{\|F'\|_2^2 + \|F\|_2^2\} = \{\|F\|_p/\|F\|_{H^1}\}^2, \tag{4.3}$$

$$M_p = \sup\{T(F) | F \in H^1, F \ne 0\}. \tag{4.4}$$

Then M_p is finite and there exists a maximizing $F \in \mathrm{SD} \cap H^1$, i.e., $T(F) = M_p$. This F is unique (up to a constant and to translation). With $r = 2/(p-2)$,

$$F(x) = (\text{const.})\,\{\cosh(x/r)\}^{-r}, \tag{4.5}$$

$$M_p = \{(2r+1)\Gamma(2r)/r\Gamma(r)^2\}^{1-2/p}(r/4)^{2/p}(r+1)^{-1}. \tag{4.6}$$

Proof. By Lemma 4.1, $T(F^*) \ge T(F)$, so henceforth we can restrict attention to $F \in \mathrm{SD}$. Then $F \in L^\infty$ since $F(x) \to 0$ as $x \to -\infty$ and $F(x)^2 = 2\int_{-\infty}^x F'(y)F(y)\,dy \le 2\|F'\|_2\|F\|_2$. Let $\{F_n\}$ be a maximizing sequence for T and we can assume $\|F_n\|_2^2 + \|F_n'\|_2^2 = 1$. By (2.9), $F_n(x) \le C|x|^{-1/2}$. By the L^∞ bound just given, $F_n(x) \le C$. Therefore,

$$F_n(x) \le h(x) \equiv \min(C, C|x|^{-1/2}) \in L^p$$

since $p > 2$. As in Theorem 2.3, we can assume $F_n \to F \in \mathrm{SD}$ pointwise. We can also assume (Banach-Alaoglu theorem) that $F_n' \to G'$ and $F_n \to G$ weakly in L^2. Clearly, $G = F$. Then

$$\liminf \|F_n'\|_2^2 + \|F_n\|_2^2 \ge \|F'\|_2^2 + \|F\|_2^2.$$

It remains to show that $M_p = \lim\|F_n\|_p = \|F\|_p$, which will also prove the crucial fact that $F \ne 0$. This follows by dominated convergence since $F_n(x) \le h(x)$.

This maximizing F can easily be found as follows: By letting $F \to F + \varepsilon\varphi$, $\varphi \in C_0^\infty$, and equating the derivative at $\varepsilon = 0$ to zero,

$$F'' = F - F^{p-1}/M_p \tag{4.7}$$

in the distributional sense. By standard ODE methods, there is only one solution to (4.5) that vanishes as $|x| \to \infty$. (Recall that $F(x) = F(-x)$ and $\|F\|_p = 1$.) This solution is (4.5), (4.6). □

It should be noted that the last step—the calculation of F and M_p—was very easy compared to the proof of Theorem 3.1. Here, it is easy to verify that (4.5) is the (essentially) unique positive solution to (4.7). In Theorem 3.1, on the other hand, it was difficult to verify that (3.1) is the desired maximizing solution to (3.9); the apparatus of stereographic projection had to be used.

Next we turn to the problem posed by (4.1) and (4.2).

THEOREM 4.3. *Let $n \geq 3$ and let $f \in W^{1,2}(\mathbf{R}^n) = H^1(\mathbf{R}^n)$. Let $1 - n/2 < b < 1$ and $p = 2n/(2b + n - 2)$, so that $\infty > p > 2$. Let*

$$R(f) = \| |x|^{-b} f \|_p / \| \nabla f \|_2, \qquad f \neq 0, \tag{4.8}$$

$$K_{n,p} = \sup\{ R(f) | f \in H^1, f \neq 0 \}. \tag{4.9}$$

(i) *If $1 > b \geq 0$, $K_{n,p}$ is finite and a maximizing $f \in \mathrm{SD}$ exists, i.e., $R(f) = K_{n,p}$.*

$$f(x) = \{1 + |x|^{2t/r}\}^{-r}, \tag{4.10}$$

$$K_{n,p} = \sigma_n^{-1/2+1/p} t^{-1/2-1/p} M_p^{1/2}, \tag{4.11}$$

with $r = 2/(p-2)$, $t = -1 + n/2$, M_p in (4.6), σ_n in (2.13).

$$K_{n,p} = [\pi n(n-2)]^{-1/2} [\Gamma(n)/\Gamma(n/2)]^{1/n}, \qquad \text{when } b = 0.$$

(ii) *If $1 - n/2 < b < 0$, $R(f)$ is unbounded on H^1, but $R(f)$ restricted to spherically symmetric functions (not necessarily decreasing) in H^1 (denoted by H^1_R) is bounded. If $K^R_{n,p} = \sup\{ R(f) | f \in H^1_R, f \neq 0\}$ then there is a maximizing $f \in \mathrm{SD}$, $R(f) = K^R_{n,p}$, given by (4.10) and $K^R_{n,p}$ is given by the right side of (4.11).*

Note. When $n > 4$, the f in (4.10) is in $H^1(\mathbf{R}^n)$. When $n = 3$ or 4, this $f \notin H^1(\mathbf{R}^n)$ but $R(f)$ is well defined.

Proof. (i) Since $|x|^{-b} \in \mathrm{SD}$, we have that $\| |x|^{-b} f \|_p \leq \| |x|^{-b} f^* \|_p$. By Lemma 4.1, $\| \nabla f \|_2 \geq \| \nabla f^* \|_2$. Thus, we can henceforth restrict our attention to $f \in \mathrm{SD}$. As in (2.11), let $F: \mathbf{R} \to \mathbf{R}$ be given by

$$F(tu) = e^{tu} f(e^u) \tag{4.12}$$

with $t = -1 + n/2 > 0$. Then

$$(\sigma_n/t)^{1/p} \| F \|_p = \| |x|^{-b} f \|_p, \tag{4.13}$$

$$(\sigma_n t)^{1/2} \| (F' - F) \|_2 = \| \nabla f \|_2 \tag{4.14}$$

where σ_n is given by (2.13). Since $f \in L^2$, as in (2.9) we have $F(u) \leq Ce^{-u/t}$.

Now assume $f \in L^\infty$ in addition to $f \in H^1 \cap \mathrm{SD}$, whence $F(u) \leq C\exp(-|u|/2t)$. Then $\int F'F = 0$, and thus

$$R(f)^2 = \sigma_n^{-1+2/p} t^{-1-2/p} T(F), \tag{4.15}$$

with $T(F)$ given by (4.3). Since $\| \nabla f \|_2 < \infty$, $F \in H^1(\mathbf{R}^1)$. Thus, for $f \in L^\infty$, Theorem 4.2 completes the proof. (Note that (4.5) and (4.12) are consistent with $f \in L^\infty$.) For $f \notin L^\infty$ we use the fact that $L^\infty \cap H^1$ is dense in H^1. Thus, there exists a sequence $\{g_n\}$ in $L^\infty \cap H^1$ such that $\|\nabla g_n\|_2 \to \|\nabla f\|_2$ and $\|g_n\|_2 \to \|f\|_2$. By passing to a subsequence, $g_n \to f$ pointwise almost everywhere and hence, $\||x|^{-b} f\|_p \le \liminf \||x|^{-b} g_n\|_p$. Therefore,

$$R(f) \le \sup\{R(f) | f \in L^\infty \cap H^1, f \neq 0\}.$$

(ii) If $b < 0$ we cannot say that we can restrict our attention to $f \in \text{SD}$. But for $f \in H^1_R(\mathbf{R}^n)$ we can make the same change of variables as in (4.12)–(4.14). The proof proceeds essentially as before. □

It is worth remarking about Theorem 4.3 as $b \to 1$ and $p \to 2$. From (4.11) we are led to believe that $K_{n,2}$ is finite, but that there is no maximizing f since (4.10) tends to unity as $p \to 2$. This is indeed correct (see [19, Lemma 2.7] where the authors attribute the result to Karlson [18] and to Herbst [17]), and $K_{n,2}$ is given by the limit of (4.11) as $p \to 2$, namely

$$(-1 + n/2)\||x|^{-1} f\|_2 \le \|\nabla f\|_2, \qquad n \ge 3, \tag{4.16}$$

and this constant is the best possible.

Another remark concerns the relation of Theorem 4.3 with $b = 0$ and Corollary 3.2. With $p = 2n/(n-2)$, $\|f\|_p \le K_{n,p}\|\nabla f\|_2 = K_{n,p}\|(-\Delta)^{1/2} f\|_2$. Formally, this is equivalent to $\|(-\Delta)^{-1/2} g\|_p \le K_{n,p}\|g\|_2$. But

$$2\pi^{(n+1)/2}(-\Delta)^{-1/2} g = \Gamma(n/2 - 1/2)|x|^{-\lambda} * g$$

with $\lambda = n - 1$, [30, p. 117]. Thus, we should have

$$2\pi^{(n+1)/2} K_{n,2n/(n-2)} = \Gamma(n/2 - 1/2) N_{2,n-1,n}, \tag{4.17}$$

which is confirmed by (3.4) and (4.11).

V. Doubly weighted HLS inequality and weighted Young inequality

Two more illustrations will be given of the use of the methods of Section II. The first is the *doubly weighted Hardy-Littlewood-Sobolev inequality* [15], [29] which generalizes the HLS inequality considered before.

THEOREM 5.1. *Let* $0 < \lambda < n$, $1 < p \le q < \infty$, $0 \le \alpha < n/p'$ (*with* $1/p + 1/p' = 1$), $0 \le \beta < n/q$ *and* $1/p + (\lambda + \alpha + \beta)/n = 1 + 1/q$. *Let*

$$V(x, y) = |x|^{-\beta}|x - y|^{-\lambda}|y|^{-\alpha} \tag{5.1}$$

be an integral kernel on $\mathbf{R}^n$. *Then* $f \to Vf$, $(Vf)(x) = \int V(x, y) f(y)\, dy$, *is a* *bounded map from* $L^p(\mathbf{R}^n)$ *to* $L^q(\mathbf{R}^n)$. *Moreover, if* $p < q$,

$$R(f) = \|Vf\|_q / \|f\|_p, \qquad f \neq 0, \quad \textit{and} \tag{5.2}$$

$$P_{\alpha,\beta,p,\lambda,n} = \sup\{R(f) | f \in L^p(\mathbf{R}^n), f \neq 0\} \tag{5.3}$$

then there is a maximizing $f \in \mathrm{SD} \cap L^p$, *i.e.*, $R(f) = P_{\alpha,\beta,p,\lambda,n}$.

Remarks. (i) In [29] the condition $0 \leq \alpha, \beta$ is relaxed to $\alpha + \beta \geq 0$. However, the stronger condition is needed here in order to use rearrangement inequalities.

(ii) Obviously, (5.2), (5.3) are equivalent to

$$P_{\alpha,\beta,p,\lambda,n} = \sup \int\int g(x)|x-y|^{-\lambda} f(y)\, dx\, dy / \||x|^\alpha f\|_p \||x|^\beta g\|_{q'}. \tag{5.4}$$

(iii) When $p = q$ a maximizing f cannot be expected to exist. See the remark at the end of Section IV which corresponds to the case $p = q = 2$, $\lambda = n - 1$, $\beta = 1$, $\alpha = 0$, $n \geq 3$. See also [17].

An extension of Lemma 2.4 is needed.

LEMMA 5.2. *Let the hypothesis be the same as in Theorem* 5.1 *except that the condition* $0 \leq \alpha, \beta$ *is eliminated. Let* $f \in L^p(\mathbf{R}^n)$ *be spherically symmetric and* $|f(r)| \leq \varepsilon r^{-n/p}$ *for all* $r > 0$. *Then* $\|Vf\|_q \leq C_n \|f\|_p^{p/q} \varepsilon^{1-p/q}$ *for some* $C_{n,\alpha,\beta}$ *independent of* f *and* ε.

Proof. This is the same as the proof of Lemma 2.4 except that (2.15) changes to

$$L_{n,\alpha,\beta}(u) = 2^{-\lambda/2} \exp\{u(n/q - \lambda/2 - \beta)\} Z_n(u). \tag{5.5}$$

The hypothesis guarantees that $|n/q - \lambda/2 - \beta| < \lambda/2$ so that $L_{n,\alpha,\beta} \in L^1(\mathbf{R})$. □

Proof of Theorem 5.1. Since $|x|^{-\alpha}$ and $|x|^{-\beta} \in \mathrm{SD}$ (here we use the fact that $\alpha, \beta \geq 0$) the generalization of the Riesz inequality given in [7] implies that a maximizing sequence $\{f_j\}$ can be taken in SD. Lemma 5.2 implies that $R(f)$ is bounded if we take $\|f_j\|_p = 1$ so that $f_j(r) \leq Cr^{-n/p}$ as in (2.9). As in (2.10), we can assume $f_j(r) \to f(r) \leq Cr^{-n/p}$ almost everywhere. If $q > p$ we can use Lemma 5.1 to dilate each f_j so that $f \neq 0$ (see the remarks after the proof of Theorem (2.3)(iv)). The final step is as in the conclusion of the proof of Theorem (2.3)(i), using Lemma 2.7. □

The second illustration is what A. Sokal has called *the weighted Young inequality*. Let f: $\mathbf{R}^n \to \mathbf{R}^+$ and let

$$f^{(m)} = f * f * \cdots * f \qquad (m \text{ factors}) \tag{5.6}$$

be the convolution of f with itself m times. We consider $m \geq 3$. Now $f^m(0)$ makes sense, even if f is defined only almost everywhere, because

$$(5.7)\qquad f^{(m)}(0) = \int f(-x_{m-1}) f(x_{m-1} - x_{m-2}) \cdots f(x_2 - x_1) f(x_1)\, dx_1 \cdots dx_{m-1}.$$

Let p and γ satisfy the conditions

$$(5.8)\qquad m/(m-1) \leq p \leq m \quad \text{and} \quad \gamma/n + 1/p = (m-1)/m \geq 0.$$

Our interest will be in the ratio

$$(5.9)\qquad R(f) = |f^{(m)}(0)| / \| |x|^\gamma f \|_p^m, \qquad f \neq 0,$$

$$(5.10)\qquad Q_{p,m,n} = \sup\{ R(f) \,|\, |x|^\gamma f \in L^p(\mathbf{R}^n), f \neq 0 \}.$$

By the generalization of the Riesz rearrangement inequality in [7], $f^{(m)}(x) \leq (f^*)^{(m)}(0)$ and $\| |x|^\gamma f^* \|_p \leq \| |x|^\gamma f \|_p$. Thus,

$$(5.11)\qquad Q_{p,m,n} = \sup\left\{ \|f^{(m)}\|_\infty / \| |x|^\gamma f \|_p^m \;\middle|\; |x|^\gamma f \in L^p(\mathbf{R}^n), f \neq 0 \right\}.$$

The idea that $R(f)$ should be bounded was suggested to me by A. Sokal. Initially, he was interested in the case $p = 2$, $m = 4$, $\gamma = n/4$ for use in a problem in quantum field theory [27]. As will shortly be seen, the $p = 2$ case reduces to the HLS inequality itself (with $p = 2$). But that is a case for which the sharp constant was derived in Section III, and thus we shall be able to compute Q when $p = 2$. Another case for which Q can be found is $p = m/(m-1)$ and this is given in (5.12).

THEOREM 5.3. *Assuming* (5.8), $Q_{p,m,n} < \infty$ *for all* p, n *and* $m \geq 3$. *Moreover, if* $m/(m-1) \leq p < m$, *there is a maximizing function,* $f \in$ SD, *i.e.,* $R(f) = Q_{p,m,n}$.

Remarks. (i) When $\gamma = 0$, $p = m/(m-1)$, (5.9) is one of the generalized Young inequalities treated in [6]. The ordinary Young inequality shows that $R(f)$ is bounded. In [6] it was shown that a maximizing f exists and that it is a Gaussian, $f(x) = \exp(-|x|^2)$. Then $Q_{p,m,n}$ can be easily computed in this case:

$$(5.12)\qquad Q_{p,m,n} = \left\{ p^{m-1}/m \right\}^{n/2}, \qquad p = m/(m-1).$$

An alternative derivation [6] of (5.12) can be obtained from the sharp constants in the ordinary Young inequality, which was also derived in [4]. If that inequality is iterated $(m-2)$ times, one obtains $Q_{p,m,n} \leq$ (right side of (5.12)). However, the explicit choice of a Gaussian for f gives a lower bound which establishes (5.12).

(ii) A generalization of Theorem 5.3 (at least the first part) is obviously possible, namely

$$(f_1 * f_2 * \cdots * f_m)(0) \le C \prod_{j=1}^{m} \| |x|^{\gamma_j} f_j \|_{p_j},$$

$\Sigma_{j=1}^m \gamma_j + n/p_j = n(m-1)$ and $\gamma_j \ge 0$, $p_j \le m$, for all j. This can easily be proved by imitating the following proof.

Proof. Let $\{f_j\}$ be a maximizing sequence and let $g_j(x) = |x|^\gamma f_j(x)$. The denominator in $R(f_j)$ is $\|g_j\|_p^m$. Then $f_j^{(m)}(0)$ is an integral over a product of $g_j(x_i - x_k)$ and $|x_i - x_k|^{-\gamma}$ factors. By the general rearrangement inequality in [7], the numerator does not decrease if we replace g_j by g_j^*. Henceforth, assume $g_j = g_j^*$ and $\|g_j\|_p = 1$. As in (2.10), we can assume $g_j(r) \to g(r) < Cr^{-n/p}$ almost everywhere and $g_j(r) < Cr^{-n/p}$. Let $g_j(r) = r^{-n/p} h_j(r)$, so that $\|h_j\|_\infty < C$ and $A_j \equiv \int h_j(x)^p |x|^{-n}\, dx < C$.

First we show that $R(f_j)$ is bounded. Substitute $f_j(x) = h_j(x)|x|^{-\gamma - n/p}$ in (5.7). By Hölder's inequality, $f_j^m(0) \le \Pi_{k=1}^m I_k(h_j)^{1/m}$ where $I_k(h_j)$ is (5.7) with $h_j(x)^m |x|^{-\gamma - n/p}$ in the kth position and $|x|^{-\gamma - n/p}$ in the other $(m-1)$ positions. It is easy to do the trivial integrals and one finds that all I_k have the common value $C\int h_j(x)^m |x|^{-n}\, dx \le CA_j \|h_j\|_\infty^{m-p}$. This shows not only that $f_j^m(0)$ is bounded but it also shows that when $p < m$, $\|h_j\|_\infty$ cannot go to zero as $j \to \infty$. Therefore there are dilations of g_j so that $g \ne 0$ (see the remarks after the proof of Theorem (2.3)(iv)).

It remains to show that $f(r) = r^{-\gamma} g(r)$ maximizes. Write $f_j = f + b_j$ with $b_j \to 0$ almost everywhere. I claim that

$$f_j^{(m)}(0) = f^{(m)}(0) + b_j^{(m)}(0) + o(1) \tag{5.13}$$

when $p < m$. This will complete the proof (using Lemma 2.6) by the strategy of Lemma 2.7. One merely sets $p/q = p/m$ in the last part of the proof of Lemma 2.7.

To prove (5.13), we have to show that when $f + b_j$ is inserted in (5.7) and expanded out into 2^m terms, those terms that contain at least one f and one b_j factor vanish as $j \to \infty$. Write $f = r^{-(\gamma + n/p)}\varphi$ and $b_j = r^{-(\gamma + n/p)}\beta_j$. We shall use a Hölder inequality as in the proof that $f^{(m)}(0)$ is bounded, but with a slight change. Consider a term, I, in $f_j^m(0)$ that has φ m_1 times and β_j m_2 times with $m_1 + m_2 = m$ and $0 < m_1 < m$. All orderings of these functions give the same integral (by changing variables). Let $a = (m-p)/(m-1) > 0$. Then $I \le I_1^{m_1/m} I_2^{m_2/m}$ where I_1 has φ^p once, φ^a $(m_1 - 1)$ times and β_j^a m_2 times. I_2 has φ^a m_1 times, β_j^p once and β_j^a $(m_2 - 1)$ times. First consider I_1. We know that φ and

β_j are bounded by a constant C (since $f_j r^{(n/p+\gamma)}$ is so bounded). Suppose the integration variable of φ^p is x_1. Then do all the other $(m-2)$ integrations and call the result $z_j(x_1)$. If we replace all the other $(m-1)$ functions by C, the $(m-2)$ integrals are finite for all $x_1 \neq 0$, namely $|x_1|^{-n}$. Therefore, by dominated convergence, $z_j(x_1) \to 0$ as $j \to \infty$ for every $x_1 \neq 0$. (Note: It is important that there is at least one factor of β_j^a and that $a > 0$.) Furthermore, $z_j(x_1)$ has the form $|x_1|^{-n} w_j(x_1)$ with $w_j(x_1)$ uniformly bounded (in x_1 and j). Thus, the final integral is $I_1 = \int \varphi(x_1)^p |x_1|^{-n} w_j(x_1)\, dx_1$. Since $|x|^{-n}\varphi(x)^p \in L^1$ and $w_j(x) < C$, $I_1 \to 0$ as $j \to \infty$ by dominated convergence. I_2 is uniformly bounded since

$$I_2 \leq C \int \beta_j(x)^p |x|^{-n}\, dx \leq C \|g_j\|_p^p = C.$$

Therefore $I \to 0$ as $j \to \infty$ and (5.13) is proved. □

The value of $Q_{p,m,n}$ has already been given in (5.12) when $p = m/(m-1)$. Let us conclude by evaluating Q when $p = 2$. The function $g(x) = |x|^\gamma f(x)$ is in L^2. Let G be the Fourier transform of g. Then $\|g\|_2 = (2\pi)^{-n/2}\|G\|_2$. The Fourier transform of $|x|^{-\gamma}$ is $w_\gamma(k)$ in (3.8) and has the form $E_{\gamma,n}|k|^{\gamma-n}$. Thus, the Fourier transform of f is $F = (2\pi)^{-n} E_{\gamma,n}|k|^{\gamma-n} * G$ and $f^{(m)}(0) = (2\pi)^{-n}\int F(k)^m\, dk$. Hence

$$(5.14) \qquad R(f)^{1/m} = (2\pi)^{-n(1/2+1/m)} E_{\gamma,n} \left| \int [|k|^{\gamma-n} * G]^m \right|^{1/m} \Big/ \|G\|_2.$$

Comparing this with (2.1) (with $p = 2$, $q = m$, $\lambda = n - \gamma$ and $f \to G$), we see that apart from a constant, the two expressions are almost the same. The one difference is that $\|\, |k|^{-\lambda} * G\|_m$ is replaced by the integral in (5.14). However, the maximizing f for (2.1) is non-negative by Corollary 3.2(ii). It can be used as G in (5.14) and the two expressions are then the same. The maximizing G is thus

$$(5.15) \qquad G = |k|^{\gamma-n} * (1 + |k|^2)^{-\gamma-n/2}.$$

This is *unique* (up to dilations, etc.).

This paper is thus brought full circle by the identification of the HLS problem and Theorem 5.3 for $p = 2$. *The maximizing f for* (5.9) *is unique* (*up to dilations, etc.*) *and is*

$$(5.16) \qquad f(x) = |x|^{-\gamma} K_\gamma(|x|), \qquad p = 2,$$

where K_γ is a Bessel function (see (3.11)). Equation (5.16) should be compared with the $p = m/(m-1)$ case in which f is a Gaussian (see (5.12)). Q can be computed from (5.14), (3.4) and (3.8).

$$(5.17) \qquad Q_{2,m,n} = \pi^{n(m-2)/4} \left\{ \frac{\Gamma(n/m)}{\Gamma(n - n/m)} \right\}^{m/2} \left\{ \frac{\Gamma(n)}{\Gamma(n/2)} \right\}^{m/2-1}.$$

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