# Fast approximate counting of linear extensions

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# The Problem

## Ranking objects

- Suppose $A, B, C, D$ each have a number
- Can compare any pair to find out which has the lower number
- Suppose data is $A < C$, $B < C$, and $B < D$

## Linear extensions

- A *linear extension* is a ranking consistent with data
- Five linear extensions for example above:

$$ABCD, ABDC, BACD, BADC, BDAC$$

- Estimate that $A$ is ranked first is $2/5$

# Counting linear extensions is hard

### How hard is it?

- Let $\Omega_{LE}$ denote the set of linear extensions
- Finding $\#\Omega_{LE}$ is a #P complete problem
- (Counting version of NP complete problem)

# Naive Acceptance/Rejection approach

To estimate $\#\Omega_{LE}$:

**Repeat** $N$ **times**

- Draw $X$ uniformly from $\Omega_{LE}$

**Output** $N/(\#$ **of times** $X$ **is** $ABCD)$

# Why is this not a good idea

## Suppose have $n$ objects to rank

- $\Omega_{LE}$ can be as large as $n!$ with $n$ objects to rank
- Need $N \approx n!$ to have significant chance of hitting $ABCD$ at all!

## More formally

- To get relative error $1 + \epsilon$ with probability at least $1 - \delta$, need

$$N = O\left(\#\Omega_{LE} n^3 (\ln n) \frac{1}{\epsilon^2} \ln\left(\frac{1}{\delta}\right)\right)$$

- Today, new algorithm

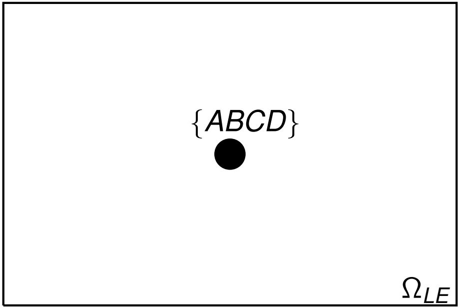
$$N = O\left((\ln \#\Omega_{LE})^2 \frac{1}{\epsilon^2} n^3 (\ln n) \ln\left(\frac{1}{\delta}\right)\right)$$

# To make this work

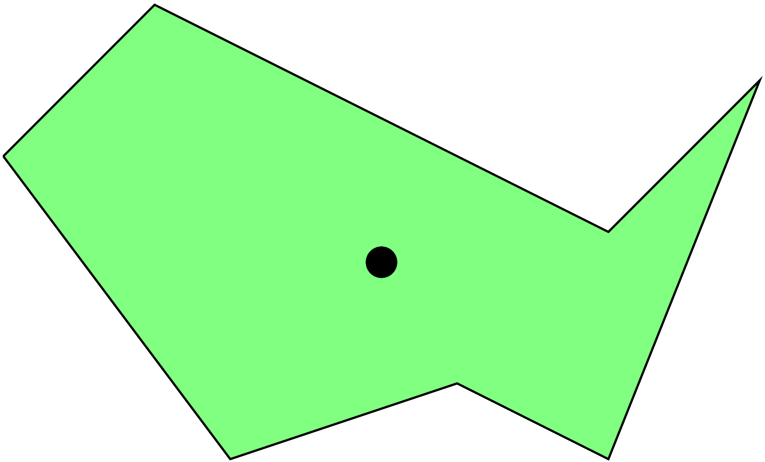
**Ingredients:**

- TPA: better than accept/reject for problems with continuous parameter
- Method for adding continuous parameter to discrete state space
- Yields new distribution on $\Omega_{LE}$ that is not uniform
- New sampling method for this nonuniform distribution

# Linear extensions, counting = discrete problem

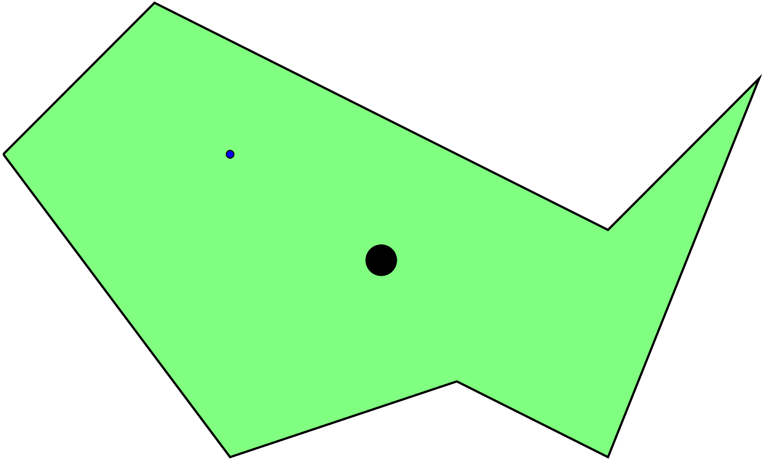


# Area = continuous problem



Black circle in center area 1, want area larger region

# Area = continuous problem



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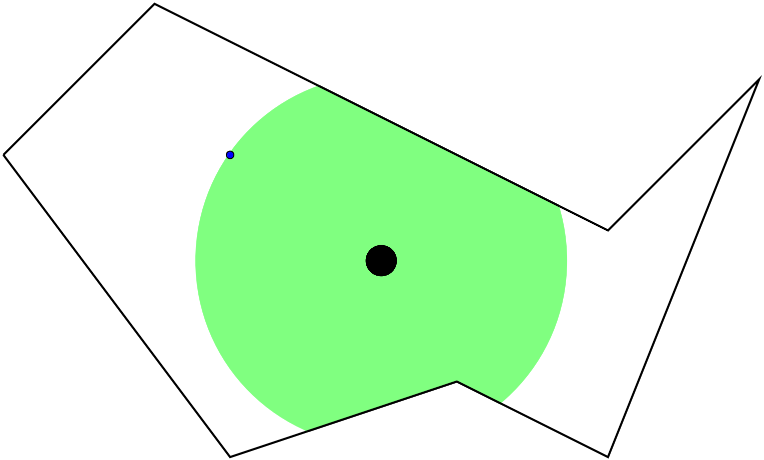

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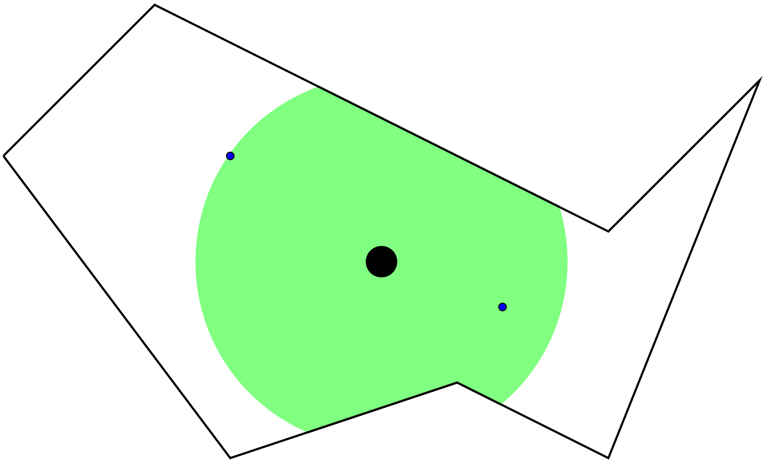

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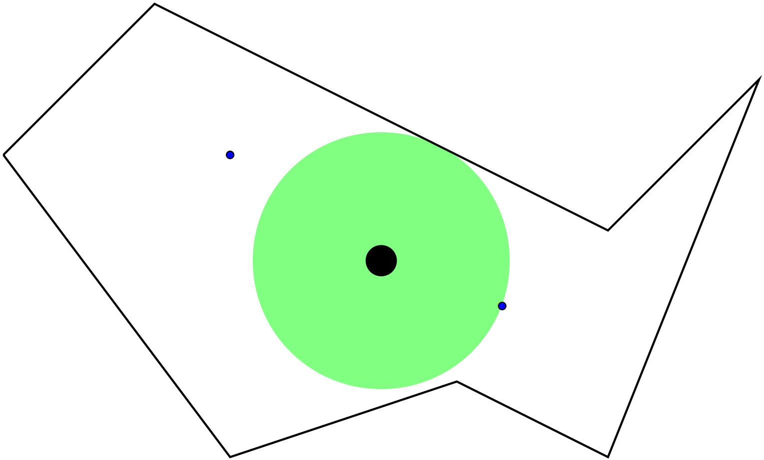

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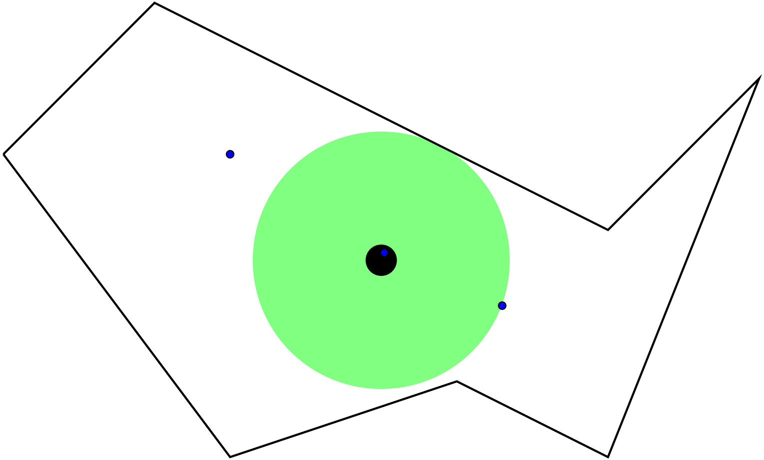

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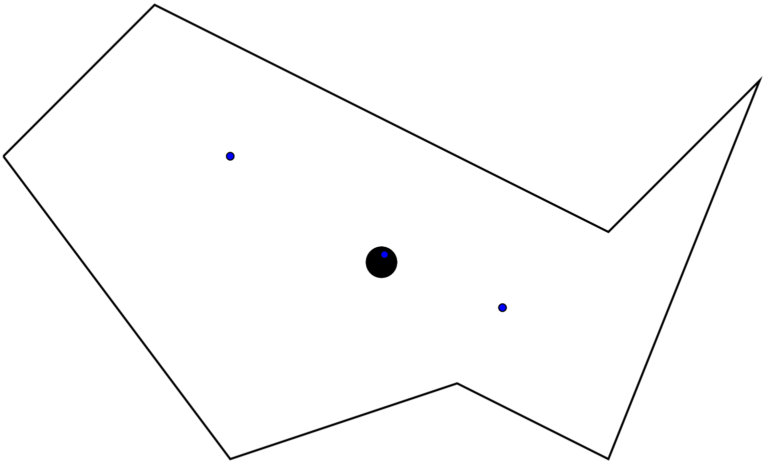

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# How many steps before reaching center?

### Steps to reach center is a random variable

- Previous example, it was 2
- Distribution is Poisson(ln(size of large region))

### Proof idea

- Shaves off uniform amount each step
- Absolute value of natural log of uniform on $[0, 1]$ is exponential (mean 1)
- So in log space, we are taking exponential steps
- Gives Poisson process
- Number of steps before reaching center also Poisson

# TPA = Tootsie Pop Algorithm

**What is a Tootsie Pop?**

- Hard candy lollipops with a tootsie roll (chewy chocolate) at the center

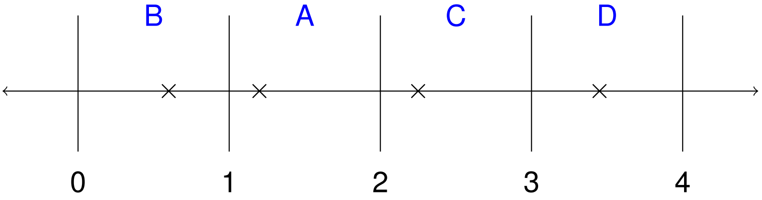
**In 1970, Mr. Owl was asked the question:**

- How many licks does it take to get to the center of a Tootsie Pop?

# Continuizing Linear Extensions

## Linear extensions are a discrete set

- They don't have a continuous parameter!
- Solution: embed in a continuous setting
- Example:

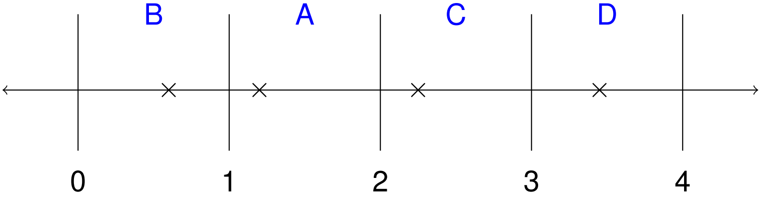



Vector: (1.21,0.62,2.25,3.45)

# Adding a continuous parameter

## One way to add parameter

- Create a home position (example: *ABCD*)
- Let $\beta$ be the farthest to the right any object is from home




$\beta = .21$ (comes from *A*)

# With parameter

$$\beta = 4, \;\; \text{set} = \Omega_{LE}$$

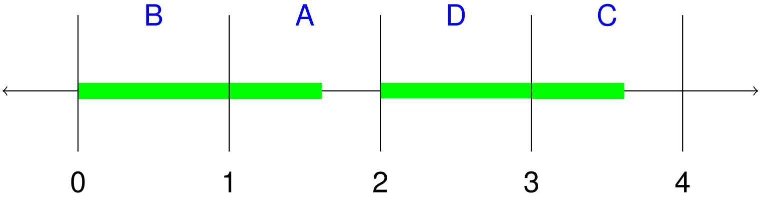
$$\beta = 0, \;\; \text{set} = \{ABCD\}$$

To use TPA, must be able to sample from any $\beta$

# The nonuniform distribution

**How does $\beta$ change the distribution on linear extensions?**

- Example: $\beta = .61$ consider *BADC*
- *A* and *C* do not have full freedom, *B* and *D* do




Volume is (1)(.61)(1)(.61)

- General weight is $\beta$ raised to the number of objects exactly $1 + \lfloor \beta \rfloor$ to the right of home

# Transposition Markov chain

**Can we sample from this distribution?**

- Use Markov chain Monte Carlo
- Pick object unfiormly, swap with one to right
- Only accept move with probability $\beta$ if moves object to $1 + \lfloor \beta \rfloor$ away from home and doesn't violate ranking data

**Does this chain mix rapidly?**

- Same technique as in [1] shows it does
- After $O(n^3 \ln n)$ steps, sample close to distribution

# Perfect sampling via bounding chains

**Even better!**

- Can draw exactly from this distribution using perfect sampling
- Coupling from the Past idea of Propp and Wilson...
- ...with bounding chains [1]
- (Can't do this with all Markov chains)
- Result: $O(n^3 \ln n)$ to get samples exactly from right distribution

# Summary: results and future work

## Use Tootsie Pop Algorithm for linear extensions

- TPA designed for continuous problems
- This is example of TPA for discrete
- Many other examples can use this technique
- Currently approx. algorithm

$$O((\ln \#\Omega_{LE})^2 n^3 \ln n)$$

## Current work

- Get algorithm down to

$$O((\ln \#\Omega_{LE}) n^3 \ln n)$$

# References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Fast perfect sampling from linear extensions
Discrete Mathematics, 306, pp. 420–428, 2006

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Bayesian Statistics 9, to appear.

J. Banks, S. Garrabrant, M. Huber and A. Perizzolo
Using TPA for approximating the number of linear extensions
Submitted.