A new algorithm for approximation of the number of linear extensions of a poset

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Example

Example (Who is the best tennis player?)

Jelena, Serena, Dinara, or Elena?

Unfortunately, they only have time to play 3 games

- Jelena beats Dinara
- Serena beats Dinara
- Jelena beats Elena

Who is most likely to be number 1?

A simple model

Prior distribution: All rankings equally likely
Condition on $J \preceq D$, $S \preceq D$, $J \preceq E$
Posterior distribution uniform on

- JSDE
- JSED
- JESD
- SJDE
- SJED

Posterior probability of being in first place:

- $\mathbb{P}(J = \text{first}) = 3/5, \mathbb{P}(S = \text{first}) = 2/5$
- $\mathbb{P}(D = \text{first}) = \mathbb{P}(E = \text{first}) = 0$

The problem

Posterior uniform over *linear extensions* of partial order

$$\mathbb{P}(A \text{ first}) = \frac{\# \text{ of lin. ext. } \sigma \text{ with } \sigma(1) = A}{\text{Total } \#\text{of lin. ext. } x}$$

The bad news: counting linear extensions is # P-complete[1]

Solution: develop approximation algorithms

[1] Brightwell & Winkler 1991

The results

Two main results

- A new method for random generation of linear extensions
- A new method for turning samples into counting algorithm

Improvement

- Sampler: Can be much faster on sparse partial orders
- Counting: $\Theta(n/(\ln n)^2)$ faster

Outline

1. Applications of linear extensions
   - Machine learning
   - Convex rank tests
2. Previous work
   - Approximating volume of convex bodies
   - Perfect simulation
   - The product estimator
3. The new algorithm
   - Reducing # of levels in product estimator
   - Retooling perfect sampler to handle new levels
   - Results

Partial orders

Notation:

$$\begin{aligned} [n] &= \{1, \ldots, n\} \\ P &= (\preceq, [n]) \text{ is a partial order on } [n] \end{aligned}$$

A partial order has three properties:

- Reflexive: $a \preceq a$
- Antisymmetric: $a \preceq b$ and $b \preceq a$ implies $a = b$
- Transitive: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$

Linear extensions

Definition

A *linear extension* of a partial order is a permutation σ of $[n]$ that respects the partial order, so

$$i < j \to \sigma(i) \preceq \sigma(j) \text{ or } \sigma(i), \sigma(j) \text{ unrelated}$$

Example: Suppose $A \preceq B$ then permutation must have the form:

$$\star \cdots \star A \star \cdots \star B \star \cdots \star$$

Goal: count Ω, the set of linear extensions of P

Challenges

Problem:

- counting linear extensions is #P-complete
- #P complete problems are harder than NP-complete
- unlikely to find polynomial time algorithm

Solution:

- Develop Monte Carlo approximation algorithm
- Come within factor of $1 + \epsilon$ with probability at least $1 - \delta$
- Want run time $\text{poly}(n)[\ln(1/\delta)]/\epsilon^2$

Machine learning[2]

Learning a binary relation

- Example: Tennis example–ranked items with some order
- Example query: Does A outrank B?
- Goal is to predict answers from a few queries
- Step 1: Use queries to create partial order
- Step 2: Use random linear extension to make predictions

The problem here is to sample linear extensions

[2] Goldman, Rivest, & Shapire 1989

Convex rank tests[3]

Nonparametric statistical model: all permutations equally likely

- *Rank tests* used for ordered data
- Tests can be viewed as partition of symmetric group S_n
- Find p-values by counting size of equivalence classes
- Each equivalence class is set of linear extensions for a partial order

The problem here is to count linear extensions

[3] Morton, Pachet, Shiu, Sturmfels, & Wienand 2008

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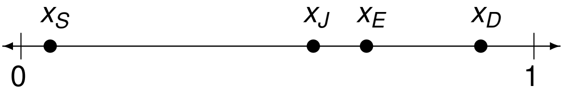
Linear extensions as convex bodies

Example: embedding linear extensions in continuous space

$$(x_S, x_j, x_e, x_d) \in [0, 1]^4$$



SJED

Enforce $A \preceq B$ with $x_A \leq x_B$

Alternate encodings

Hit and run chain

- Choose random direction
- Choose random point inside convex body

For any convex body:[4]

- Steps per sample (amortized): $O(n^3(\ln n)^3)$
- Steps for approximately counting: $O(n^4(\ln n)^7)$
- Each step: $\Theta(\#\text{relations in partial order})$
- Total time: $\Theta(n^6(\ln n)^7)$
- Impractical: $O(\cdot)$ hides large constant (at least 1000)

[4] Lovász and Vempala 2006

Bounding chains and coupling from the past (CFTP)

CFTP[5] is a perfect simulation protocol

- Draws samples exactly from uniform distribution
- Running time is a random variable
- Requires extra construction such as: monotonicity, bounding chains[6]

Bounding chain for linear extensions[7]

- Uses adjacent transposition chain
- $O(n^3 \ln n)$ steps for sparse problems

[5] Propp & Wilson 1996
[6] H. 2004
[7] H. 2006

Adjacent transposition chain

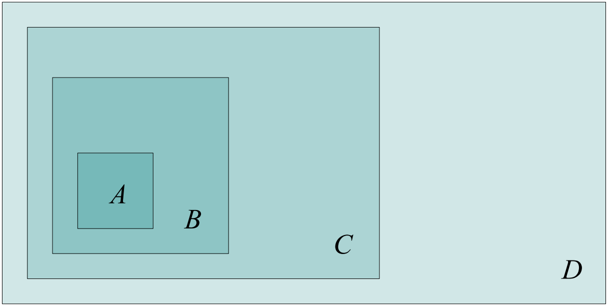
One step in chain

- Stay at same state with probability $1/2$
- Otherwise choose i uniformly from $\{1, \ldots, n\}$
  swap items at position i and $i + 1$

Examples

$$\begin{aligned} SJDE + \text{position } 1 + \text{move} &= JSDE \\ SJDE + \text{position } 3 + \text{stay} &= SJDE \\ SJDE + \text{position } 2 + \text{move} &= SJDE \text{ (since } J \preceq D) \end{aligned}$$

Product Estimator



Know: $\#A$

Want: $\#D$

$$\widehat{\#D} = \widehat{\left(\frac{\#D}{\#C}\right)} \widehat{\left(\frac{\#C}{\#B}\right)} \widehat{\left(\frac{\#B}{\#A}\right)} \#A$$

Total # of samples needed: Ck^2/ϵ^2

▸ General procedure

Naive Product Estimator

At each level *fix* one element of permutation

Example (* = wildcard)

- ****
- J***
- JS**
- JSD* = JSDE

Parameters:

- Number of levels: n
- $C = n$
- Total # of samples: $O(n^3)$

Previous work versus today

	Convex Bodies	Discrete	Today
Sampling	Hit and run $O(n^5(\ln n)^3)$	Adjacent Trans. $O(n^3 \ln n)$	Adj. Trans. Plus restrictions $O(n^3 \ln n)$
Counting	Advanced prod. est. $O(n^6(\ln n)^9)$	Naive prod. estimator $O(n^6 \ln n)$	Improved prod. est. $O(n^5(\ln n)^3)$

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Slicing state space more efficiently

Previously

- Ratio between levels: $C = n$
- Number of levels: $k = n$

Today

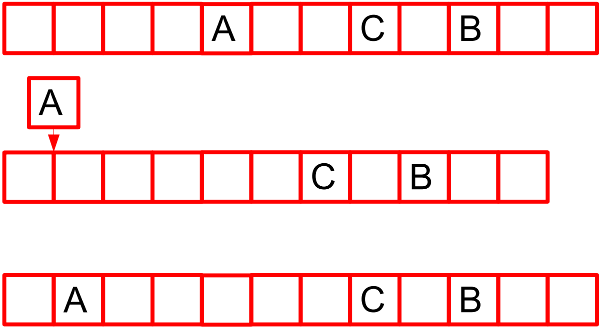
- Ratio between levels: $C = 2$
- Number of levels: $k = n \log_2 n$

Distribution of a maximal element

Stationary update function

- Remove item A from permutation
- Reinsert item A uniformly among available positions

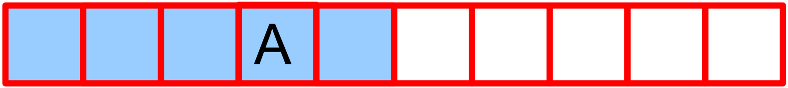
Example: $A \preceq B$, $A \preceq C$



Probability max element in left half at least 1/2

In general

- If no element precedes A
- $\mathbb{P}(\sigma^{-1}(A) \leq \lceil n/2 \rceil) \geq 1/2$



How to slice the state space in half

Restricted linear extensions

- Start with upper bound $\sigma^{-1}(A) = n_1$, $n_1 = n$
- Next $\sigma^{-1}(A) = n_2$, $n_2 = \lceil n_1/2 \rceil$
- Continues until $n_j = 1$

Example

- $A \preceq B$, $A \preceq C$
- Current state $* * * A * * B * * C$
- If A chosen, after one step A equally likely to be in first 6 positions

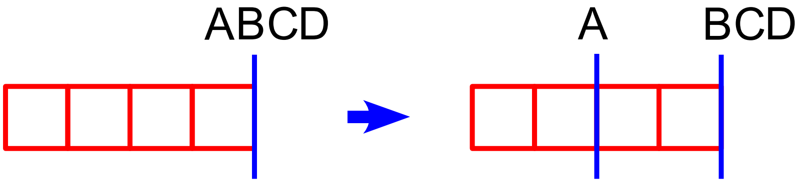
In general

- If no element precedes A
- $\mathbb{P}(\sigma^{-1}(A) \leq \lceil n/2 \rceil) \geq 1/2$

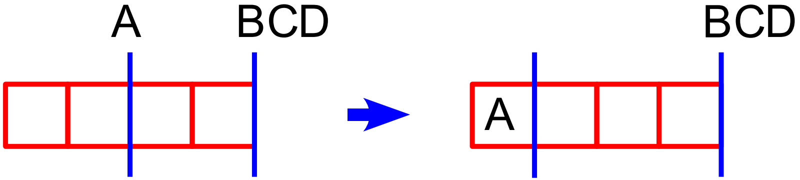
```
function n = product_estimator(A,epsilon,delta)

n = size(A); n = n(1);  % Find side length of matrix A;
k = ceil(log(n)/log(2));
est = zeros(n-1,k);
num_per_level = 4*(n - 1)*k / epsilon^2 * log(1/delta);
for move_item = 1:(n-1)
    move_from_loc = n;
    move_to = 0;
    while (move_from_loc > move_item)
        move_to = move_to + 1;
        move_to_loc = max(move_item,ceil(move_from_loc / 2));
        for samples = 1:num_per_level
           x = gler(A,move_item - 1,move_item,move_from_loc);
           est(move_item,move_to) = est(move_item,move_to) + ...
                (sum(x(1:move_to_loc) == move_item) > 0);
        end
        move_from_loc = move_to_loc;
        est(move_item,move_to)=log(est(move_item,move_to)/num_per_level);
    end
end
n = exp(-sum(sum(est)));
```

Levels with 4 states...



Levels with 4 states...



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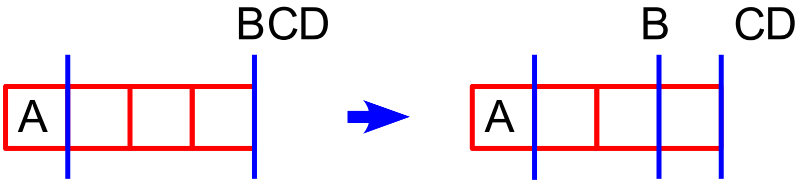

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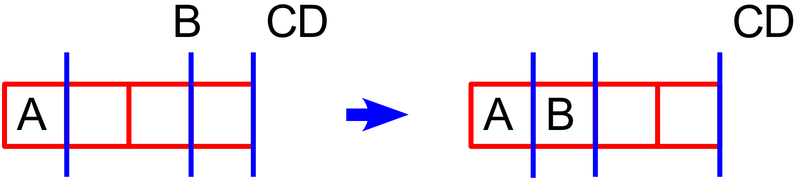
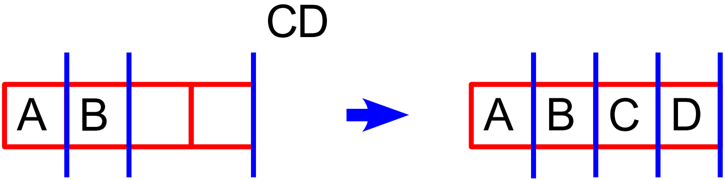


Levels with 4 states...



To generate samples

Key line in earlier code:

```
x = gler(A,move_item - 1,move_item,move_from_loc);
```

Need function `gler` to generate restricted linear extensions

- Need $A \preceq B \Rightarrow \sigma^{-1}(A) < \sigma^{-1}(B)$
- For all items α, $\sigma^{-1}(\alpha) \leq r(\alpha)$

How to get samples

Bounding chains	+	Couping From the Past	=	Perfect Simulation

Perfect simulation = draws exactly uniform over linear extensions

- Technique similar to [H. 2004]
- Bounding chain needs modification
- Running time same as for unrestricted
- Still $O(n^3 \ln n)$

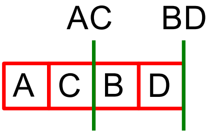
Bounding chains[8]

Bounding state tells what positions can be occupied

- Similar to levels in product estimator
- When upper bounds different, unique permutation bounded
- Modification to handle levels straightforward

Example of bounding state:

$$x = ACBD \qquad\qquad y = (2, 4, 2, 4)$$



▸ More details

[8] H. 2004, H. 2006

Further work

Other perfect simulation methods

- Monotonic Markov chain
- Embed permutations in $[0, 1]^n$
- Delete-reinsert Step 1: Choose an item
- Delete-reinsert Step 2: Remove item
- Delete-reinsert Step 3: Uniformly place item back in

Can be faster for sparse partial orders:

$$O(n \ln n) \text{ for empty } P$$

Advanced product estimator

Summary

Linear extensions arise in several statistical applications

- Counting exactly is difficult
- Fully polynomial randomized approximation scheme (fpras)

New product estimator, new perfect simulator

- Product estimator fewer levels
- Perfect simulator same speed as old
- Needs $3n^5(\ln n)^3\epsilon^{-2}\ln(1/\delta)$ expected number of uniforms
- Beats previous $1000n^6(\ln n/\epsilon^2)^9\epsilon^{-2}\ln(1/\delta)$

References

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Ann. Probab., 19(3):1367–1392, 1991.

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Convex rank tests and semigraphoids,
arXiv:math/0702564v2[math.CO], 2008.

Many ways to create convex body

Mathews[9] used

$$\Omega = \{x \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots x_d^2 \leq 1\}$$

Again, enforce partial order via half-space constraints:

$$A \preceq B \Rightarrow x_A \leq x_B$$

Mathews devised Markov chain step specialized to this space

Go back

[9] Mathews1991

General procedure

Given sets $A_1 \subset \cdots \subset A_k$

- k levels
- Create enough levels so $\#A_i/\#A_{i-1} \leq C$ for all i
- Estimate $\#A_{i-1}/\#A_i$ using N samples from A_i
- Estimate $\#A_k$ with telescoping product
- Call final estimate $\hat{p}$
- $\mathbb{E}[p] = \#A_k$
- For $\mathrm{SD}(p) = \epsilon\mathbb{E}[p]$, set $N = Ck/\epsilon^2$

Total # of samples needed: Ck^2/ϵ^2

Want to minimize C and minimize number of levels k

Go back

Bounding chains for adjacent transpositions[10]

Upper bounds positions

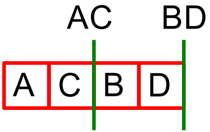
- Permuting items $A_1, \ldots, A_n$
- $x^{-1}(A_i) \leq y(A_i)$ for all items A_i
- Trick is to update (x, y) to next state (x', y') so that:

$$(\forall i)(x^{-1}(A_i) \leq y(A_i)) \to (\forall i)(x'^{-1}(A_i) \leq y'(A_i))$$

Example of bounding state:

$x = ACBD$, $x^{-1}(A) = 1, x^{-1}(B) = 3, x^{-1}(C) = 2, x^{-1}(D) = 4$

$y = (2, 4, 2, 4)$



[10] H. 2006

Bounding a single permutation

When only a single x for y:

- Say $y(i) \neq y(j)$ for all $i \neq j$
- Then $x^{-1} = y$ only state bounded by chain

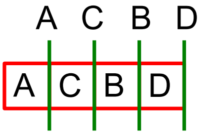
Example:

$$y = (2, 4, 2, 4)$$

$x^{-1} = (1, 3, 2, 4)$ or
$x^{-1} = (2, 3, 1, 4)$ or
$x^{-1} = (1, 4, 2, 3)$ or
$x^{-1} = (2, 4, 1, 3)$

$$y = (1, 3, 2, 4)$$

$$x^{-1} = (1, 3, 2, 4)$$



Typical run

Take a fixed number T of bounding chain steps:

$y = (4, 4, 4, 4)$
(bounds everything)

$y = (1, 3, 2, 4)$
(good outcome–one bounded state)

$y = (2, 4, 2, 4)$
(bad outcome–many bounded states)

What is it good for?

Two uses for bounding chains

- Expected time to bound single state gives an upper bound on mixing time of chain
- Together with coupling from the past, can generate samples exactly from uniform distribution

Go back