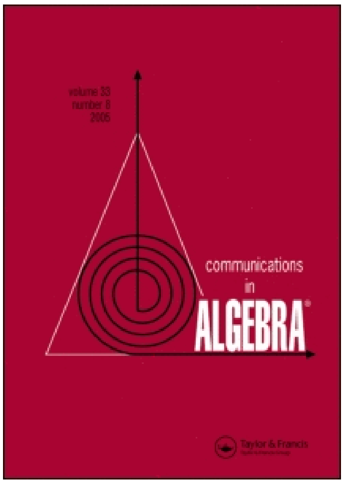

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Finite groups can be arbitrarily hamiltonian

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Finite Groups Can Be Arbitrarily Hamiltonian

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Abstract

Let r be a rational in (0,1]. There exists a finite group G for which the proportion of elements g and subgroups H satisfying $gHg^{-1} = H$ is r. An analogous result holds for three other measures of 'Hamiltonianness'.

1 Introduction

Let the finite group G act on the sets $S = S(G)$ and $C = C(G)$ of its subgroups and cyclic subgroups, respectively, by conjugation. Let $NS = NS(G)$ and $NC = NC(G)$ denote the normal and normal cyclic subgroups of G, respectively. Here are four measures of 'Hamiltonianness' for G. (Recall that a finite group is Hamiltonian if each of its subgroups is normal.)

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- $\mu(G) = P_G(S) = k(G)/|S|$: the ratio of the number of conjugacy classes in S to the number of subgroups of G. This measure is the proportion of elements g and subgroups H satisfying $gHg^{-1} = H$ (see [1]).
- $\mu(G) = P_{\overline{G}}(S) = |NS|/|S|$: the proportion of subgroups of G that are normal.
- $\mu(G) = P_G(C) = k(C)/|S|$: the ratio of the number of conjugacy classes in C to the number of cyclic subgroups of G.
- $\mu(G) = P_{\overline{G}}(C) = |NC|/|C|$: the proportion of cyclic subgroups of G that are normal.

It was shown in [1] that for each rational number $r \in (0,1]$, there exists a sequence of finite groups $\{G_n\}$ such that $\lim_{n\to\infty} \mu(G_n) = r$ where μ is any one of these measures. In fact, each rational number $r \in (0,1]$ is assumed by each of these four measures. Specifically,

Theorem *For each rational number $r \in (0,1]$, there exists a a finite group G such that $P_G(S) = k(G)/|S| = r$.*

The purpose of this paper is to provide a contructive proof of this theorem and to exhibit the appropriate construction for the other measures.

2 Proof of the theorem

If $r = 1$, then any Hamiltonian group will do. Otherwise, an appropriate group may be selected from the class

$$J(p,n) = \langle a, b \,|\, a^p = b^{2^n} = e \text{ and } bab^{-1} = a^{-1}\rangle$$

where p is an odd prime and n is a positive integer.

The following facts concerning $J(p,n)$ are clear. $|J(p,n)| = p2^n$. $J(p,n)$ has one Sylow p-subgroup and p Sylow 2-subgroups, all of which are cyclic. $J(p,n)$ has a cyclic subgroup $J_0(p,n) = \langle a, b^2\rangle$ of index two. $J_0(p,n)$ has $2n$ subgroups, all of which are characteristic in $J_0(p,n)$ and are therefore normal in $J(p,n)$. Any subgroup of $J(p,n)$ not contained in $J_0(p,n)$ contains a Sylow 2-subgroup of $J(p,n)$, so is one of the p Sylow 2-subgroups of $J(p,n)$ or $J(p,n)$ itself. There are therefore $2n+p+1$ subgroups of $J(p,n)$. Each of these subgroups, but for $J(p,n)$ itself, is cyclic and $2n$ of the subgroups are normal.

It follows that

$$\begin{aligned}
P_{J(p,n)}(S) &= \frac{2n+2}{2n+p+1},\\
P_{\overline{J(p,n)}}(S) &= \frac{2n+1}{2n+p+1},\\
P_{J(p,n)}(C) &= \frac{2n+1}{2n+p},\\
P_{\overline{J(p,n)}}(C) &= \frac{2n}{2n+p}.
\end{aligned}$$

Lemma *Let $a, b \in \mathbf{Z}^+$ be such that $a < b$. Then*

$$\frac{a}{b} = \frac{2n+2}{2n+p+1}$$

for some integer $n \geq 1$ and some odd prime p.

Proof. We find an integer m such that

$$\frac{a}{b} = \frac{2am}{2bm} = \frac{2am}{2am+p-1};$$

i.e., we find a solution m to the equation $2bm = 2am+p-1$, or equivalently $m(b-a)+1 = p$. By Dirichlet's Theorem, $m(b-a)+1 = p$ has a solution for $p \geq 3$, and $m \geq 2$. Since $2am$ is even and is greater than or equal to 4, it may be written as $2n+2$ for some suitable positve integer n. The denominator becomes $2n+p+1$ and we are done. □

Therefore, for each rational number $r \in (0,1)$, we may find p and n such that $P_{J(p,n)}(S) = r$.

3 Constructions for the other measures

The construction for each of the other measures can be summarized as follows.

i) Write $r = a/b$ as a product of ratios, each of which is is less than one. (This is technical, but elementary and in the spirit of the lemma, so is not included here.)

ii) Construct groups with pairwise relatively prime orders whose measures are the factors of a/b.

iii) Take G to be the direct product of these groups. The measure of this direct product is the product of the measures because of ii.

To complete this program we need the following two classes of groups. Let p be an odd prime and let $n \geq 3$.

$$K(p,n) = \langle a, b | a^{p^{n-1}} = b^p = e \text{ and } bab^{-1} = a^{p^{n-2}+1} \rangle$$

$$H(p,n) = K(p,n) \times \langle c | c^p = e \rangle$$

The groups $K(p,n)$ were used in [1] and it was shown there that

$$\begin{aligned} P_{\overline{K(p,n)}}(S) &= \frac{(n-2)(p+1)+3}{(n-1)(p+1)+2}, \\ P_{K(p,n)}(C) &= \frac{(n-2)p+3}{(n-1)p+2}, \\ P_{\overline{K(p,n)}}(C) &= \frac{(n-2)p+2}{(n-1)p+2}. \end{aligned}$$

For the groups $H(p,n)$ we observe that

$$P_{\overline{H(p,n)}}(S) = \frac{(n-1)[(p^2+1)+p(p+1)] - 2p^2+p+3}{(n-1)[(p^2+1)+p(p+1)]+p+3}.$$

This follows from

$$|NS(H(p,n))| = |S(H(p,n))| - 2p^2$$

and

$$|S(H(p,n))| = (n-1)[(p^2+1)+p(p+1)]+p+3.$$

The proofs of these facts, which amount to adjusting the corresponding results for $K(p,n)$ for the cyclic factor $\langle c \rangle$, are elementary (but tedious) so are not included here.

To achieve $P_{\bar{G}}(S) = a/b$, we write

$$\frac{a}{b} = \frac{53n_1 - 97}{53n_1 - 48} \cdot \frac{2n_2 + 1}{2n_2 + p + 1} \cdot \frac{4n_3 - 5}{4n_3 - 2}$$

for some integers $n_1 \geq 3, n_2 \geq 1, n_3 \geq 3$ and a prime p different from 2, 3, and 7 and choose

$$G = H(7, n_1) \times J(p, n_2) \times K(3, n_3).$$

To achieve $P_G(C) = a/b$, we write

$$\frac{a}{b} = \frac{2n_1 + 1}{2n_1 + p} \cdot \frac{3n_2 - 3}{3n_2 - 1}$$

for some integers $n_1 \geq 1$, $n_2 \geq 3$ and some prime $p \geq 5$ and choose

$$G = J(p, n_1) \times K(3, n_2).$$

To achieve $P_{\bar{G}}(C) = a/b$, we write

$$\frac{a}{b} = \frac{2n_1}{2n_1 + p} \cdot \frac{(n_2 - 2)q + 2}{(n_2 - 1)q + 2}$$

for some integers $n_1 \geq 1$ and $n_2 \geq 3$ and distinct odd primes p and q and choose

$$G = J(p, n_1) \times K(q, n_2).$$

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Reference

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