Stable image reconstruction using total variation minimization

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Introduction

Grayscale digital images do not fill the entire space of $N \times N$ blocks of pixel values, consisting primarily of slowly-varying pixel intensities except along edges. In other words, digital images are compressible with respect to their discrete gradient. Compressed sensing (CS) provides the technology to exploit compressibility when acquiring signals of general interest, allowing for accurate and robust signal acquisition from surprisingly few measurements.

Here we present near-optimal guarantees for accurate and robust image recovery from under-sampled noisy measurements using total variation minimization, and our results may be the first of this kind. In particular, we show that from $O(s \log(N))$ nonadaptive linear measurements, an image can be reconstructed to within the best $s$-term approximation of its gradient, up to a logarithmic factor.

Types of Compressibility

- a) Original $X$
- b) Haar Wavelet Transform
- c) Gradient

Horizontal and vertical discrete directional derivatives (top). The gradient of the image may often be more sparse than its Haar wavelet transform (bottom).

Main Results

Theorem 1. Let $N = 2^n$. Let $A(X) : C^{N \times N} \rightarrow C^n$ be a linear operator described by matrices $A_j$ satisfying the restricted isometry property (RIP) of order $s$ and level $\delta < 1/3$.

If $X \in C^{N \times N}$ has discrete gradient $\nabla X$ and noisy measurements $y = A(X) + \xi$ are observed with noise level $\|\xi\|_2 \leq \epsilon$, then

$$\hat{X} = \arg\min_z \|Z\|_{TV} \text{ such that } \|A(Z) - y\|_2 \leq \epsilon$$

satisfies

$$\|\nabla X - \hat{X}\|_2 \leq \frac{\|\nabla X - \nabla \hat{X}\|_1}{\sqrt{s}} + \epsilon,$$

and

$$\|X - \hat{X}\|_2 \leq \frac{\log(N^2/s)}{\sqrt{s}} \left( \frac{\|\nabla X - \nabla \hat{X}\|_1}{\sqrt{s}} + \epsilon \right).$$

To our best knowledge, Theorem 1 provides the first provable guarantee of robust recovery for images from compressed measurements via total variation minimization.

Optimality. The gradient error guarantees (1) and (2) provided by the theorem are optimal, and the image error guarantee (3) is optimal up to a logarithmic factor, which we conjecture to be an artifact of the proof.

RIP Requirements. The RIP requirements in Theorem 1 mean that the linear measurements can be generated from standard RIP matrix ensembles. For example, they can be generated from a subgaussian random matrix $\Phi \in R^{m \times N}$ with $m \geq s \log(N^2/s)$ or a partial random Fourier matrix $F \in C^{m \times N}$ with $m \geq s \log^2(N)$, admitting a $O(N^2 \log(N))$ matrix-vector multiply.

Side Length. Theorem 1 requires the image side-length to be a power of 2, $N = 2^n$. This is not actually a restriction, as an image of side-length $N \in N$ can be reflected horizontally and vertically to produce a $2N \times 2N$ image with the same total-variation up to a factor of 4.

Theorem 2 (Strong Sobolev inequality). Let $B : C^{N \times N} \rightarrow C^n$ be a linear map such that $B \circ H^{-1} : C^{N \times N} \rightarrow C^m$ has the RIP of order $2s$ and level $\delta < 1$, where $H$ is the biharmonic Haar wavelet transform. Suppose that $D \in C^{N \times N}$ satisfies the tube constraint $\|B(D)\|_2 \leq \epsilon$. Then

$$\|D\|_2 \leq \left( \frac{\|B(D)\|_{TV}}{\sqrt{s}} \right) \log(N^2/s) + \epsilon.$$

Open Questions

We recently extended our results to the multidimensional case, for signals with higher dimensional structure such as movies. We conjecture however that the logarithmic factor in (3) is an artifact of the proof and not necessary. Further, the measurement operator $A$ is derived from RIP matrices, but has a specific structure (see [2]), whose modification we also leave these items for future work.

References