We consider here a homogeneous system of \( n \) first order linear equations with constant, real coefficients:

\[
\begin{align*}
    x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
    x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\
    &\vdots \\
    x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n
\end{align*}
\]

This system can be written as \( \mathbf{x}' = \mathbf{Ax} \), where

\[
\mathbf{x}(t) = \begin{pmatrix}
    x_1(t) \\
    x_2(t) \\
    \vdots \\
    x_m(t)
\end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]
Equilibrium Solutions

- Note that if $n = 1$, then the system reduces to
  \[ x' = ax \quad \Rightarrow \quad x(t) = e^{at} \]

- Recall that $x = 0$ is the only equilibrium solution if $a \neq 0$.

- Further, $x = 0$ is an asymptotically stable solution if $a < 0$, since other solutions approach $x = 0$ in this case.

- Also, $x = 0$ is an unstable solution if $a > 0$, since other solutions depart from $x = 0$ in this case.

- For $n > 1$, equilibrium solutions are similarly found by solving $Ax = 0$. We assume $\det A \neq 0$, so that $x = 0$ is the only solution. Determining whether $x = 0$ is asymptotically stable or unstable is an important question here as well.
**Phase Plane**

- When $n = 2$, then the system reduces to
  
  \[ x'_1 = a_{11}x_1 + a_{12}x_2 \]
  
  \[ x'_2 = a_{21}x_1 + a_{22}x_2 \]

- This case can be visualized in the $x_1x_2$-plane, which is called the **phase plane**.

- In the phase plane, a direction field can be obtained by evaluating $Ax$ at many points and plotting the resulting vectors, which will be tangent to solution vectors.

- A plot that shows representative solution trajectories is called a **phase portrait**.

- Examples of phase planes, directions fields and phase portraits will be given later in this section.
Solving Homogeneous System

To construct a general solution to $x' = Ax$, assume a solution of the form $x = \xi e^{rt}$, where the exponent $r$ and the constant vector $\xi$ are to be determined.

Substituting $x = \xi e^{rt}$ into $x' = Ax$, we obtain

$$r\xi e^{rt} = A\xi e^{rt} \iff r\xi = A\xi \iff (A - rI)\xi = 0$$

Thus to solve the homogeneous system of differential equations $x' = Ax$, we must find the eigenvalues and eigenvectors of $A$.

Therefore $x = \xi e^{rt}$ is a solution of $x' = Ax$ provided that $r$ is an eigenvalue and $\xi$ is an eigenvector of the coefficient matrix $A$. 
Example 1: Direction Field  (1 of 9)

Consider the homogeneous equation $x' = Ax$ below.

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$

A direction field for this system is given below.

Substituting $x = \xi e^{rt}$ in for $x$, and rewriting system as $(A-rI)\xi = 0$, we obtain

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
Example 1: Eigenvalues (2 of 9)

Our solution has the form \( \mathbf{x} = \xi e^{rt} \), where \( r \) and \( \xi \) are found by solving

\[
\begin{pmatrix}
1-r & 1 \\
4 & 1-r
\end{pmatrix}
\begin{pmatrix}
\xi \\
\xi_1
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Recalling that this is an eigenvalue problem, we determine \( r \) by solving \( \det(\mathbf{A} - r\mathbf{I}) = 0 \):

\[
\begin{vmatrix}
1-r & 1 \\
4 & 1-r
\end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r - 3)(r + 1)
\]

Thus \( r_1 = 3 \) and \( r_2 = -1 \).
Example 1: First Eigenvector (3 of 9)

Eigenvector for $r_1 = 3$: Solve

\[(A - rI)\xi = 0 \iff \begin{pmatrix} 1 & -3 & 1 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\]

by row reducing the augmented matrix:

\[
\begin{pmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \xi_1 & -1/2 \xi_2 = 0 \\ 0 \xi_2 = 0 \end{pmatrix}
\]

\[
\rightarrow \xi^{(1)} = \begin{pmatrix} 1/2 \xi_2 \\ \xi_2 \end{pmatrix} = c \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \ c \text{ arbitrary} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
Example 1: Second Eigenvector (4 of 9)

Eigenvector for $r_2 = -1$: Solve

$$ (A - rI)\xi = 0 \iff \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} $$

by row reducing the augmented matrix:

$$ \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 0 \\ 4 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow 1\xi_1 + 1/2\xi_2 = 0 $$

$$ 0\xi_2 = 0 $$

$$ \xi^{(2)} = \begin{pmatrix} -1/2\xi_2 \\ \xi_2 \end{pmatrix} = c \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \ c \text{ arbitrary} \rightarrow \text{choose } \xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} $$
Example 1: General Solution (5 of 9)

The corresponding solutions $x = \xi e^{rt}$ of $x' = Ax$ are

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad x^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

The Wronskian of these two solutions is

$$W[x^{(1)}, x^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{-2t} \neq 0$$

Thus $x^{(1)}$ and $x^{(2)}$ are fundamental solutions, and the general solution of $x' = Ax$ is

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$
Example 1: Phase Plane for $\mathbf{x}^{(1)}$  

To visualize solution, consider first $\mathbf{x} = c_1\mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \iff x_1 = c_1 e^{3t}, \ x_2 = 2c_1 e^{3t}$$

Now

$$x_1 = c_1 e^{3t}, \ x_2 = 2c_1 e^{3t} \iff e^{3t} = \frac{x_1}{c_1} = \frac{x_2}{2c_1} \iff x_2 = 2x_1$$

Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2x_1$, which is the line through origin in direction of first eigenvector $\mathbf{\xi}^{(1)}$

If solution is trajectory of particle, with position given by $(x_1, x_2)$, then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.  

In either case, particle moves away from origin as $t$ increases.
Example 1: Phase Plane for $\mathbf{x}^{(2)}$ (7 of 9)

Next, consider $\mathbf{x} = c_2 \mathbf{x}^{(2)}$:

$$
\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \quad \Leftrightarrow \quad x_1 = c_2 e^{-t}, \quad x_2 = -2c_2 e^{-t}
$$

Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2x_1$, which is the line through origin in direction of 2nd eigenvector $\mathbf{\xi}^{(2)}$.

If solution is trajectory of particle, with position given by $(x_1, x_2)$, then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.

In either case, particle moves towards origin as $t$ increases.
Example 1:  
Phase Plane for General Solution  

The general solution is $x = c_1 x^{(1)} + c_2 x^{(2)}$:

$$x(t) = c_1 \left( \begin{array}{c} 1 \\ 2 \end{array} \right) e^{3t} + c_2 \left( \begin{array}{c} 1 \\ -2 \end{array} \right) e^{-t}$$

As $t \to \infty$, $c_1 x^{(1)}$ is dominant and $c_2 x^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach the line $x_2 = 2x_1$ as $t \to \infty$.

Similarly, for $c_2 \neq 0$, all solutions asymptotically approach the line $x_2 = -2x_1$ as $t \to -\infty$.

The origin is a **saddle point**, and is unstable. See graph.
Example 1:
Time Plots for General Solution  

The general solution is \( \mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} \):

\[
\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \iff \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix}
\]

As an alternative to phase plane plots, we can graph \( x_1 \) or \( x_2 \) as a function of \( t \). A few plots of \( x_1 \) are given below.

Note that when \( c_1 = 0 \), \( x_1(t) = c_2 e^{-t} \to 0 \) as \( t \to \infty \). Otherwise, \( x_1(t) = c_1 e^{3t} + c_2 e^{-t} \) grows unbounded as \( t \to \infty \).

Graphs of \( x_2 \) are similarly obtained.
Example 2: Direction Field  (1 of 9)

Consider the homogeneous equation \( x' = Ax \) below.

\[
x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x
\]

A direction field for this system is given below.

Substituting \( x = \xi e^{rt} \) in for \( x \), and rewriting system as \( (A-rI)\xi = 0 \), we obtain

\[
\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
Example 2: Eigenvalues (2 of 9)

Our solution has the form \( x = \xi e^{rt} \), where \( r \) and \( \xi \) are found by solving

\[
\begin{pmatrix}
-3 - r & \sqrt{2} \\
\sqrt{2} & -2 - r
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_1
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Recalling that this is an eigenvalue problem, we determine \( r \) by solving \( \det(A - rI) = 0 \):

\[
\left| \begin{array}{cc}
-3 - r & \sqrt{2} \\
\sqrt{2} & -2 - r
\end{array} \right| = (-3 - r)(-2 - r) - 2 = r^2 + 5r + 4 = (r + 1)(r + 4)
\]

Thus \( r_1 = -1 \) and \( r_2 = -4 \).
Eigenvector for $r_1 = -1$: Solve

$$\begin{align*}
(A - rI) \xi &= 0 \\
\begin{pmatrix} -3 + 1 & \sqrt{2} \\ \sqrt{2} & -2 + 1 \end{pmatrix} &\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\iff \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}$$

by row reducing the augmented matrix:

$$
\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \iff \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \iff \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$\xi^{(1)} = \begin{pmatrix} \sqrt{2}/2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$
Example 2: Second Eigenvector

Eigenvector for \( r_2 = -4 \): Solve

\[
(A - rI)\xi = 0 \iff \begin{pmatrix} -3 + 4 & \sqrt{2} \\ \sqrt{2} & -2 + 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

by row reducing the augmented matrix:

\[
\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(2)} = \begin{pmatrix} -\sqrt{2} \xi_2 \\ \xi_2 \end{pmatrix}
\]

choose \( \xi^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \)
Example 2: General Solution (5 of 9)

The corresponding solutions $x = \xi e^{rt}$ of $x' = Ax$ are

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad x^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

The Wronskian of these two solutions is

$$W[x^{(1)}, x^{(2)}](t) = \begin{vmatrix} e^{-t} & -\sqrt{2} e^{-4t} \\ \sqrt{2} e^{-t} & e^{-4t} \end{vmatrix} = 3e^{-5t} \neq 0$$

Thus $x^{(1)}$ and $x^{(2)}$ are fundamental solutions, and the general solution of $x' = Ax$ is

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$
Example 2: Phase Plane for $\mathbf{x}^{(1)}$ (6 of 9)

To visualize solution, consider first $\mathbf{x} = c_1 \mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} \quad \Leftrightarrow \quad x_1 = c_1 e^{-t}, \quad x_2 = \sqrt{2} c_1 e^{-t}$$

Now

$$x_1 = c_1 e^{-t}, \quad x_2 = \sqrt{2} c_1 e^{-t} \quad \Leftrightarrow \quad e^{-t} = \frac{x_1}{c_1} = \frac{x_2}{\sqrt{2} c_1} \quad \Leftrightarrow \quad x_2 = \sqrt{2} x_1$$

Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2^{1/2} x_1$, which is the line through origin in direction of first eigenvector $\xi^{(1)}$.

If solution is trajectory of particle, with position given by $(x_1, x_2)$, then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.

In either case, particle moves towards origin as $t$ increases.
Example 2: Phase Plane for \( x^{(2)} \) (7 of 9)

Next, consider \( x = c_2 x^{(2)} \):

\[
x^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \iff x_1 = -\sqrt{2} c_2 e^{-4t}, \ x_2 = c_2 e^{-4t}
\]

Then \( x^{(2)} \) lies along the straight line \( x_2 = -2^{\frac{1}{2}} x_1 \), which is the line through origin in direction of 2nd eigenvector \( \xi^{(2)} \)

If solution is trajectory of particle, with position given by \( (x_1, x_2) \), then it is in Q4 when \( c_2 > 0 \), and in Q2 when \( c_2 < 0 \).

In either case, particle moves towards origin as \( t \) increases.
Example 2: 
Phase Plane for General Solution (8 of 9)

The general solution is \( \mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} \):

\[
\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}
\]

As \( t \to \infty \), \( c_1 \mathbf{x}^{(1)} \) is dominant and \( c_2 \mathbf{x}^{(2)} \) becomes negligible. Thus, for \( c_1 \neq 0 \), all solutions asymptotically approach origin along the line \( x_2 = \frac{1}{2} x_1 \) as \( t \to \infty \).

Similarly, all solutions are unbounded as \( t \to -\infty \).

The origin is a node, and is asymptotically stable.
Example 2:
Time Plots for General Solution  (9 of 9)

The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} - \sqrt{2} c_2 e^{-4t} \\ \sqrt{2} c_1 e^{-t} + c_2 e^{-4t} \end{pmatrix}$$

As an alternative to phase plane plots, we can graph $x_1$ or $x_2$ as a function of $t$. A few plots of $x_1$ are given below.

Graphs of $x_2$ are similarly obtained.
2 x 2 Case: 
Real Eigenvalues, Saddle Points and Nodes

The previous two examples demonstrate the two main cases for a 2 x 2 real system with real and different eigenvalues:

- Both eigenvalues have opposite signs, in which case origin is a saddle point and is unstable.
- Both eigenvalues have the same sign, in which case origin is a node, and is asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.
Eigenvalues, Eigenvectors and Fundamental Solutions

In general, for an $n \times n$ real linear system $x' = Ax$:

- All eigenvalues are real and different from each other.
- Some eigenvalues occur in complex conjugate pairs.
- Some eigenvalues are repeated.

If eigenvalues $r_1, \ldots, r_n$ are real & different, then there are $n$ corresponding linearly independent eigenvectors $\xi^{(1)}, \ldots, \xi^{(n)}$. The associated solutions of $x' = Ax$ are

$$x^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \ldots, x^{(n)}(t) = \xi^{(n)} e^{r_n t}$$

Using Wronskian, it can be shown that these solutions are linearly independent, and hence form a fundamental set of solutions. Thus general solution is

$$x = c_1 \xi^{(1)} e^{r_1 t} + \ldots + c_n \xi^{(n)} e^{r_n t}$$
Hermitian Case: Eigenvalues, Eigenvectors & Fundamental Solutions

If $A$ is an $n \times n$ Hermitian matrix (real and symmetric), then all eigenvalues $r_1, \ldots, r_n$ are real, although some may repeat.

In any case, there are $n$ corresponding linearly independent and orthogonal eigenvectors $\xi^{(1)}, \ldots, \xi^{(n)}$. The associated solutions of $x' = Ax$ are

$$x^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \ldots, x^{(n)}(t) = \xi^{(n)} e^{r_n t}$$

and form a fundamental set of solutions.
Example 3: Hermitian Matrix  (1 of 3)

Consider the homogeneous equation $x' = Ax$ below.

$$x' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x$$

The eigenvalues were found previously in Ch 7.3, and were:

$r_1 = 2$, $r_2 = -1$ and $r_3 = -1$.

Corresponding eigenvectors:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$
Example 3: General Solution (2 of 3)

The fundamental solutions are

\[ x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \quad x^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \quad x^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t} \]

with general solution

\[ x = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t} \]
Example 3: General Solution Behavior  (3 of 3)

The general solution is $x = c_1x^{(1)} + c_2x^{(2)} + c_3x^{(3)}$:

$$x = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

As $t \to \infty$, $c_1x^{(1)}$ is dominant and $c_2x^{(2)}$, $c_3x^{(3)}$ become negligible.

Thus, for $c_1 \neq 0$, all solns $x$ become unbounded as $t \to \infty$, while for $c_1 = 0$, all solns $x \to 0$ as $t \to \infty$.

The initial points that cause $c_1 = 0$ are those that lie in plane determined by $\xi^{(2)}$ and $\xi^{(3)}$. Thus solutions that start in this plane approach origin as $t \to \infty$. 
Complex Eigenvalues and Fundamental Solns

If some of the eigenvalues \( r_1, \ldots, r_n \) occur in complex conjugate pairs, but otherwise are different, then there are still \( n \) corresponding linearly independent solutions

\[
\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \xi^{(n)} e^{r_n t},
\]

which form a fundamental set of solutions. Some may be complex-valued, but real-valued solutions may be derived from them. This situation will be examined in Ch 7.6.

If the coefficient matrix \( \mathbf{A} \) is complex, then complex eigenvalues need not occur in conjugate pairs, but solutions will still have the above form (if the eigenvalues are distinct) and these solutions may be complex-valued.
Repeated Eigenvalues and Fundamental Solns

If some of the eigenvalues \( r_1, \ldots, r_n \) are repeated, then there may not be \( n \) corresponding linearly independent solutions of the form

\[ x^{(1)}(t) = \xi^{(1)} e^{rt}, \ldots, x^{(n)}(t) = \xi^{(n)} e^{rt} \]

In order to obtain a fundamental set of solutions, it may be necessary to seek additional solutions of another form.

This situation is analogous to that for an \( n \)th order linear equation with constant coefficients, in which case a repeated root gave rise solutions of the form

\[ e^{rt}, te^{rt}, t^2 e^{rt}, \ldots \]

This case of repeated eigenvalues is examined in Section 7.8.